NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

A MATHEMATICAL MODEL FOR CALCULATING
DETECTION PROBABILITY
OF A DIFFUSION TARGET

by

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September 1984

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The primary objective of this study is to derive a mathematical model to predict the detection probability of a target which moves randomly, according to a two-dimensional diffusion model. This model assumes that there is a stationary searcher which has a "cookie-cutter" sensor with radius R. In order to construct this model, a Monte Carlo simulation program is used to generate detection probabilities. It is demonstrated

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that this model can be used asymptotically to predict an upper bound detection probability of an "equivalent" random tour target.

A Mathematical Model for Calculating Detection Probability of a Diffusion Target

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ABSTRACT

The primary objective of this study is to derive a mathematical model to predict the detection probability of a target which moves randomly, according to a two-dimensional diffusion model. This model assumes that there is a stationary searcher which has a "cookie-cutter" sensor with radius R. In order to construct this model, a Monte Carlo simulation program is used to generate detection probabilities. It is demonstrated that this model can be used asymptotically to predict an upper bound detection probability of an "equivalent" random tour target.

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I. DESCRIPTION OF THE DIFFUSION MODEL

A. INTRODUCTION

The main objective of this thesis is to find and test an experimental mathematical model which predicts the probability of detecting a two-dimensional target by a stationary searcher. This model will be shown to provide an upper bound for the probability of detection by a stationary searcher of a target conducting a "random tour" [Ref. 1]

B. DESCRIPTION OF DIFFUSION MODEL

1. The Searcher Location

The searcher is assumed to be located in the center of a square search region of area A. This location is held fixed during the search period. The searcher has a detection capability over a disk of radius R. The detection probability of a target inside of this disk is 1 and outside is 0. The searcher thus has a "cookie-cutter" sensor with detection range R. [Ref. 2]

2. The Target Starting Position

The target's starting position is uniformly distributed over the square search region A.

3. Motion of the Target

In our diffusion model, the target moves randomly over the area A as a diffusing particle which reflects off the area boundaries. The diffusion constant is D, which has dimensions of area per unit of time. In any time interval of length Δt that does not contain a boundary reflection,

components of the target's position on the X and Y axes suffer increments which are independent of all previous increments and which are each distributed normally with mean 0 and variance $D\Delta t$.

Still ignoring boundary effects, the diffusion assumption results in the target's location at time t having a circular bivariate normal probability distribution with mean of the starting position and variance of Dt.

Thus the probability density of the target's location at time t is

$$f(x,y,t) = \frac{1}{2\pi Dt} \exp\left(-\frac{(x-U_x)^2 + (y-U_y)^2}{2Dt}\right) \qquad (1.1)$$

where (U_x, U_y) is the target's starting position. Adding the effects of boundary reflection significantly complicates the calculation of f(x,y,t) and leads to the necessity of using simulation to attack this problem.

4. Detection

Detection occurs whenever the target enters the searcher circular detection disk which has a radius R.

C. DIFFUSION SIMULATION MODEL (DIFSIM)

A Monte Carlo simulation computer model (DIFSIM) is used to genarate detection probabilities for this diffusion model. This program is written in FORTRAN and designed for use at the Naval Postgraduate School(NPGS). It uses the new version of the NPGS Random Number Genarator Package, called LLRANDOMII in order to generate Uniform and Normal random numbers.

1. Inputs

- Area size, A, in square nautical miles.
- Diffusion constant, D, in square nautical miles per hour.
- Radius of detection disk, R, in nautical miles.
- Number of replications.
- Detection period as an hour.
- Time increment, Δt, for each discrete step in minutes.

2. Functioning of Program

The initial target position is selected from a bivariate uniform distribution over the search region A. Subsequent target positions are determined by a discrete approximation of the diffusion. We make the following definitions,

X=x component of current location

Y=y component of current location

X'=x component of new location at the end of time increment Δt

Y'=y component of new location at the end of time increment Δt

Then,

$$X' = X + \theta_x (D\Delta t)^{1/2}$$

$$Y' = Y + \theta_y (D\Delta t)^{1/2}$$

where θ_x and θ_y are drawn independently from a standard normal distribution.

In this model a 5 minute Δt is used. Different time increments, varying from 1 minute up to 15 minutes, have

been tested and 5 minutes has been accepted as a good value. For smaller time increments the simulation program took too long in computer execution time. As shown in Figure 1.1 there is no significant difference in probability curves between 5 minute increments and smaller time increments.

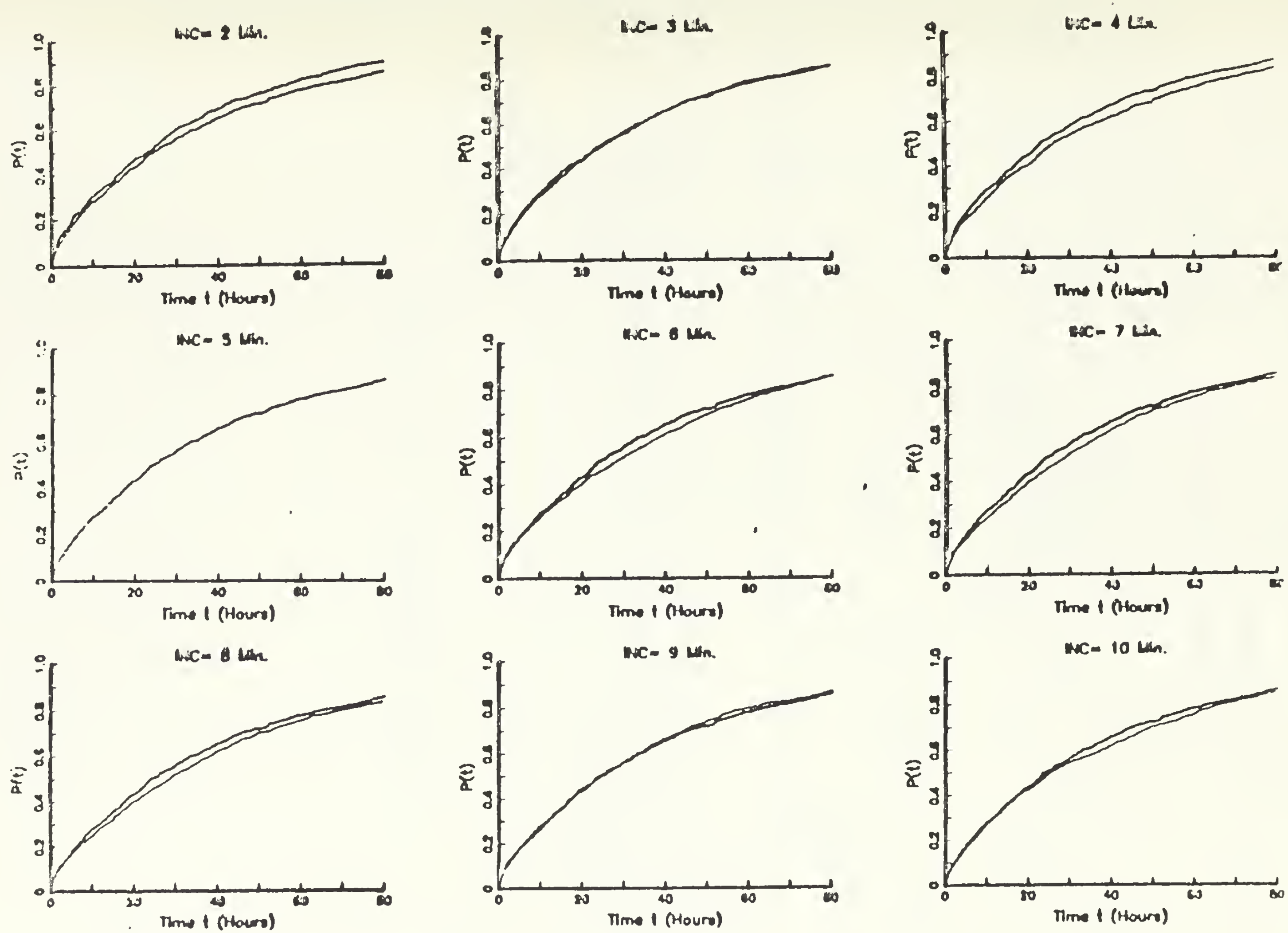


Figure 1.1 P(t) (Probability of Det. by Time t) for Various Time Increments.

When the target encounters a boundary, a reflection is made to keep the target inside the search area. The target's Y' position after a reflection is given as follows:

$Y' < 0 \quad \Rightarrow Y'$ becomes $-Y'$

$Y' > a \quad \Rightarrow Y'$ becomes $2a-Y'$

where a is the length of a side of the square search area A. The target reflects in the X direction in a similar manner.

Detection occurs whenever the target enters the detection disk. This event can be defined analytically as follows:

$$(X-\frac{a}{2})^2 + (Y-\frac{a}{2})^2 \leq R^2$$

3. Output

For each time t, the simulation output is the ratio N_D/N_T where

N_D=number of replications giving a detection by time t,
N_T=total number of Monte Carlo Replications used in the simulation.

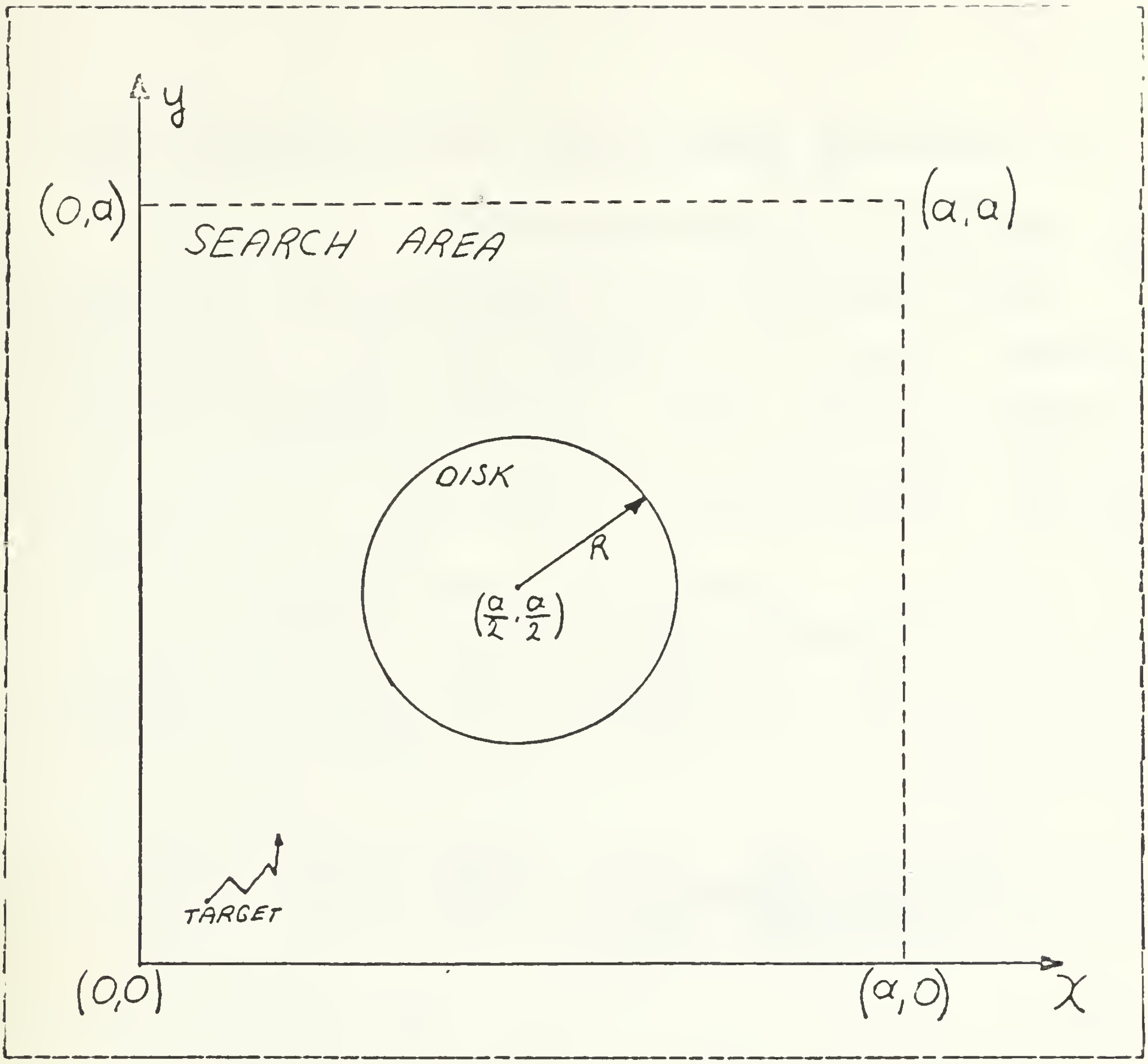


Figure 1.2 Diffusion Model.

II. CONNECTION BETWEEN RANDOM TOUR AND DIFFUSION MODEL

A. DESCRIPTION OF RANDOM TOUR MODEL

In the random tour model considered here, the target is assumed to move at a constant speed and to make course changes at random times. Each new course is drawn from uniform distribution on $[0, 2\pi]$. The lengths of the time between course changes is exponentially distributed with mean $1/\lambda$.

An analytic expression for the probability density of the target's position after a random tour of length t was derived in [Ref. 1] Given the target's initial position at the origin of a two-dimensional coordinate system, this expression is

$$g(x,y,t) = \frac{1}{2\pi(vt)^2}\left[\frac{\lambda t}{\sqrt{1-\rho^2}}\exp\left(-\lambda t\left(1-\sqrt{1-\rho^2}\right)\right) + \exp^{-\lambda t}\left(\delta(\rho-1)\right)\right] \quad (2.1)$$

where

V=target speed (nautical mile in per hour)

λ=course change rate (hrs)

t=time (hrs)

$$\rho^2 = \frac{x^2+y^2}{(vt)^2}$$

The Dirac δ_function component of g(x,y,t) arises from the fact that with probability $e^{-\lambda t}$ the target does not finish the first step by time t. In other words, the target makes no course change through time t. Therefore, its probability mass is concentrated on the boundary of a disk of radius vt and centered at the origin.

B. CONNECTION BETWEEN RANDOM TOUR AND DIFFUSION MODELS

Let R denote the radial distance of the target from the origin at time t. Then

$$R^2 = X^2 + Y^2$$

The expression for $E[R^2]$ for the random tour model was derived in [Ref. 3] as follows:

$$E[R^2] = \iint\limits_{0 \le x^2+y^2 \le (vt)^2} (x^2+y^2)\, g(x,y,t)\, dx\,dy \qquad (2.2)$$

Substituting (2.1) into (2.2) and transforming to polar coordinates, we obtain

$$E[R^2] = \int_0^{2\pi}\int_0^{Vt} r^3 \frac{e^{-\lambda t}}{2\pi (Vt)^2} \left\{ \delta\left(\frac{r}{Vt} - 1\right) + \frac{\lambda t}{\sqrt{1-\left(\frac{r}{Vt}\right)^2}} \exp\left(\lambda t \sqrt{1-\left(\frac{r}{Vt}\right)^2}\right)\right\} dr\,d\theta$$

Setting $x = r/vt$, we have

$$E[R^2] = e^{-\lambda t} (Vt)^2 \int_0^1 x^3 \left\{ \delta(x-1) + \frac{\lambda t}{\sqrt{1-\left(\frac{r}{Vt}\right)^2}} \exp\left(\lambda t \sqrt{1-x^2}\right)\right\} dx$$

$$= e^{-\lambda t} (Vt)^2 \left\{ 1 + \lambda t \int_0^1 \frac{x^3}{1-x^2} \exp\left\{\lambda t \sqrt{1-x^2}\right\} dx \right\} \qquad (2.3)$$

To perform the integration in (2.3), we set $u=\sqrt{1-x^2}$. That gives

$$\lambda t \int_0^1 \frac{x^3}{1-x^2} \exp\left\{\lambda t\sqrt{1-x^2}\right\} dx$$

$$= \lambda t \int_0^1 (1-u^2) \exp\left(\lambda t u\right) du$$

$$= e^{\lambda t} - 1 - \frac{1}{(\lambda t)^2}\int_0^{\lambda t} y^2 e^y \, dy$$

$$= \frac{2e^{\lambda t}}{(\lambda t)^2}(\lambda t - 1 + e^{-\lambda t}) - 1 \qquad (2.4)$$

From (2.3) and (2.4) we then have

$$E[R^2] = 2\,\frac{V^2}{\lambda^2}(\lambda t - 1 + e^{-\lambda t})$$

$$= \frac{2V^2 t}{\lambda}\left(1 - \frac{1-e^{-\lambda t}}{\lambda t}\right) \qquad (2.5)$$

In the calculation of $E[R^2]$ for diffusion model with diffusion constant D, the target's initial position is assumed to be at the origin of a two-dimensional coordinate system. We have

$$E[R^2] = E[X^2 + Y^2] = E[X^2] + E[Y^2] \qquad (2.6)$$

Since X and Y are independent and uncorrelated, they have normal marginal distributions. So,

$$X \sim N(0, Dt)$$

$$Y \sim N(0, Dt)$$

and

$$E[X^2] = Var[X] + (E[X])^2 = Dt + 0 = Dt$$

$$E[Y^2] = Var[Y] + (E[Y])^2 = Dt + 0 = Dt$$

If we substitute these $E[X^2]$ and $E[Y^2]$ values into (2.6), we get

$$E[R^2] = E[X^2] + E[Y^2] = Dt + Dt = 2Dt \qquad (2.7)$$

As described in [Ref. 3] for the random tour model, as t goes to infinity the Central Limit Theorem requires that g(x,y,t) becomes asymptotically circular bivariate normal with mean $\mu = 0$ and variance $\sigma^2 = Vt/\lambda$. This result can be obtained by using the formula (2.5) and letting t go to infinity.

$$\lim_{t \to \infty} E[R^2] = \lim_{t \to \infty} \frac{2V^2 t}{\lambda}\left(1 - \frac{1 - e^{-\lambda t}}{\lambda t}\right) = \frac{2V^2 t}{\lambda} \qquad (2.8)$$

By comparing the equations (2.7) and (2.8), it is seen that as t becomes large, a random tour can be approximated by a diffusion with diffusion constant

$$D = \frac{V^2}{\lambda} \qquad (2.9)$$

To examine the relationship between a random tour and its "equivalent" diffusion, two simulations were used. Example results of these two simulation programs, DIFSIM and

PASS are displayed in Figure 2.1. Both programs are Monte Carlo search simulations. For PASS (Passive Acoustic Submarine Simulation), [Ref. 4] the target motion model is a random tour. In DIFSIM (Diffusion Simulation), the target moves as a diffusing particle. In both cases, the searcher is stationary at the center of a 100 nm x 100 nm search area and has a cookie_cutter sensor with detection range 15nm. The target reflects off the area boundaries. The initial target position was selected uniformly over the search area, and each replication of the simulation was allowed to continue until the target moved within distance 15 nm of the searcher.

To generate the results shown in Figure 2.1, the following 5 different pairs of λ and V are used as a rate of course change and speed of target in the PASS simulation model.

$$\begin{pmatrix} V \\ \lambda \end{pmatrix} = \begin{pmatrix} 5.0 \\ 0.25 \end{pmatrix}, \begin{pmatrix} 10.0 \\ 1.0 \end{pmatrix}, \begin{pmatrix} 15.0 \\ 2.25 \end{pmatrix}, \begin{pmatrix} 20.0 \\ 4.0 \end{pmatrix}, \begin{pmatrix} 25.0 \\ 6.25 \end{pmatrix} \begin{pmatrix} \text{nm/hr} \\ \text{1/hr} \end{pmatrix}$$

If we use equation (2.9), we may get an "equivalent" diffusion constant 100 nm²/hr, by using each different (λ,V) pair. Thus 100 nm²/hr is used as a diffusion constant in the DIFSIM in order to get an "equivalent" diffusion model.

As demonstrated, detection probabilities are asymptotically very close to each other as t increases. But during the early search hours, detection probabilities for a diffusion model are higher than the probabilities which are generated by the random tour model.

If we recall the the equations (2.5) and (2.8) we will see that

$$E[R^2] \leq \frac{2V^2 t}{\lambda} = 2Dt$$

Thus the approximation of $E[R^2]$ for the random tour model by using the diffusion model always leads to an <u>OVERESTIMATE</u> of $E[R^2]$.

In the diffusion model we may expect that the target will move a greater distance from the origin than does the target in the "equivalent" random tour model. It is therefore reasonable to expect that the diffusing target will encounter a stationary searcher more quickly than will a target conducting an "equivalent" random tour. This conjecture has been supported by further simulation testing; also supported is the fact that the two processes are asymptotically identical.

An experimental analytical model will be constructed for the diffusion model in the next section.

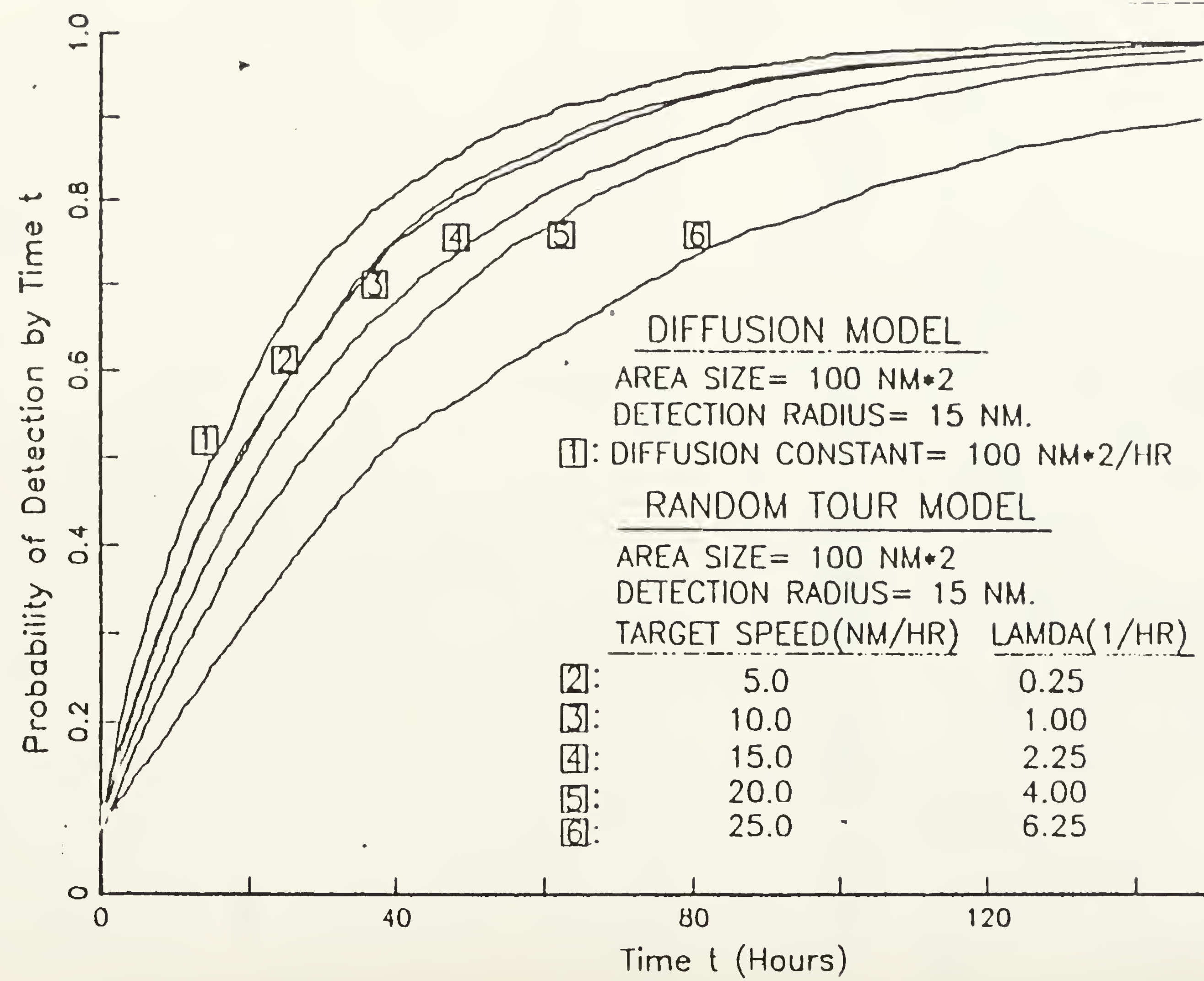


Figure 2.1 Comparision of Diffusion and Equivalent Random Tour Models.

III. ESTABLISHMENT OF THE ANALYTICAL MODEL

In this chapter, and with simulation results from DIFSIM, an experimental analytical model will be constructed for predicting the probability of a diffusion target entering a stationary disk by some time t.

A. ASSUMPTIONS

The following assumptions are used in our model

- The searcher is fixed at the center of a square search region of area A.
- The searcher has a detection capability over a disk of radius R with a detection probability of 1 within the disk and 0 outside. (i.e, a cookie-cutter detector).
- The target's starting position is uniformly distributed over a search region A.
- The target moves randomly over the area A as a diffusion particle with diffusion constant D.
- The target reflects off the area boundaries.
- A target can be detected only once by the searcher

B. CLASSIFICATION OF THE VARIABLES

The variables in our model are,

- First detection probability, P.
- Search area, A (nm^2).
- Searcher detection disk radius, R (nm).
- Target diffusion constant, D (nm^2/hr).

•Time, t (hr).

The first detection probability, P, is the dependent variable. The remaining variables A,R,D and t are independent. That is.

$$P = f(A,R,D,t)$$

C. CONSTRUCTION OF THE MODEL

Figure 3.1 shows four plots of the probability of a target detection by time t as estimated by the Monte Carlo simulation DIFSIM. If we look at these curves, we will observe that all of them have an increasing trend and they approach 1 asymptotically. It also appears as if the second derivative must be negative everywhere. Figure 3.2 plots 1 minus the same data on a log scale. The fact that these plots are very nearly linear suggests the following functional form P(t).

$$P(t)=1-\alpha e^{-\beta t} \qquad (3.1)$$

where α and β are determined by the problem parameters R,A and D. After conducting 23 separate simulations with different values of R,A and D, the author is convinced that the form of P(t) is approximately exponential. This thesis attempts to establish values of α and β, as function of R,A and D, to allow equation (3.1) to be a reasonable estimate.

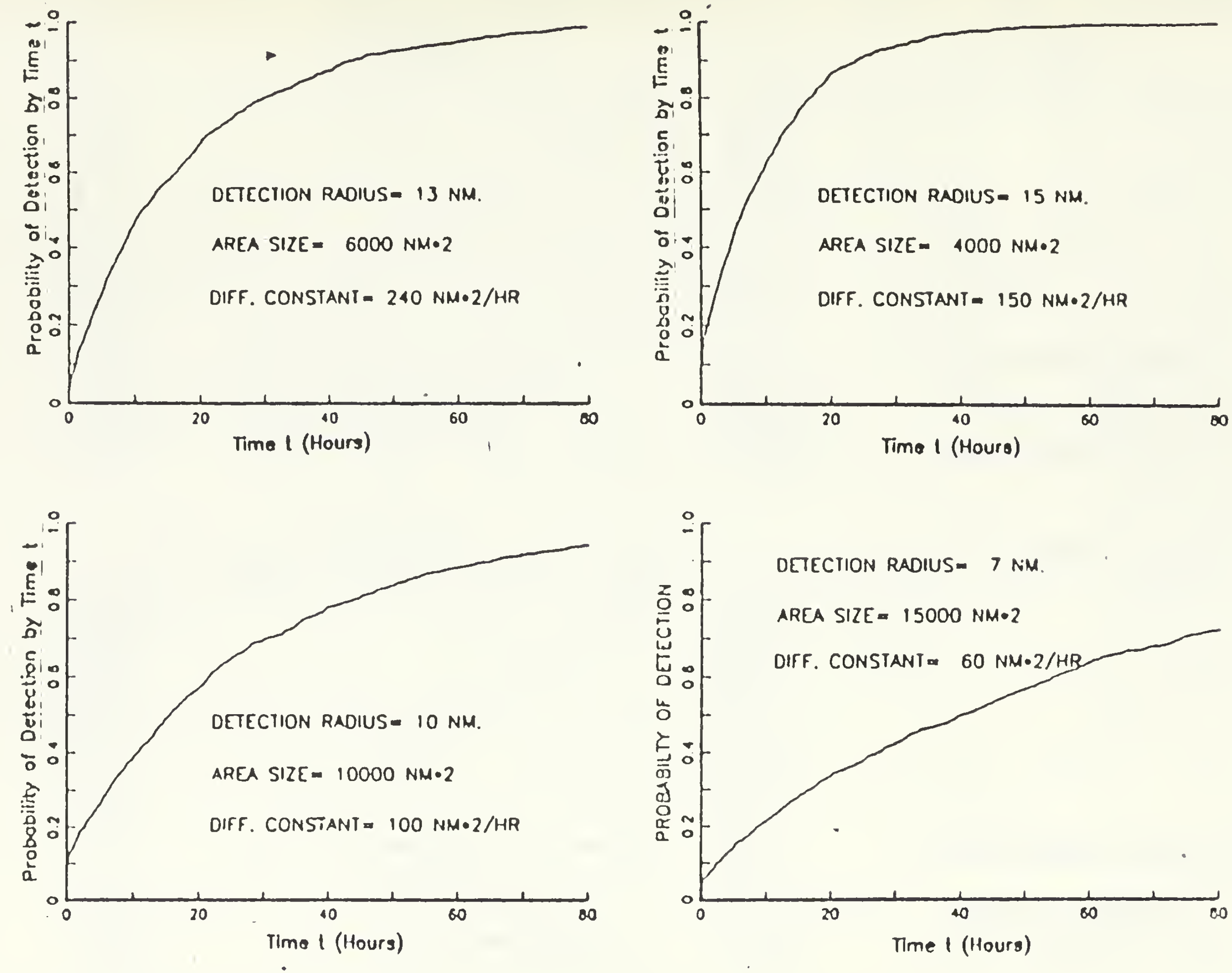


Figure 3.1 Example Plot of P(t).

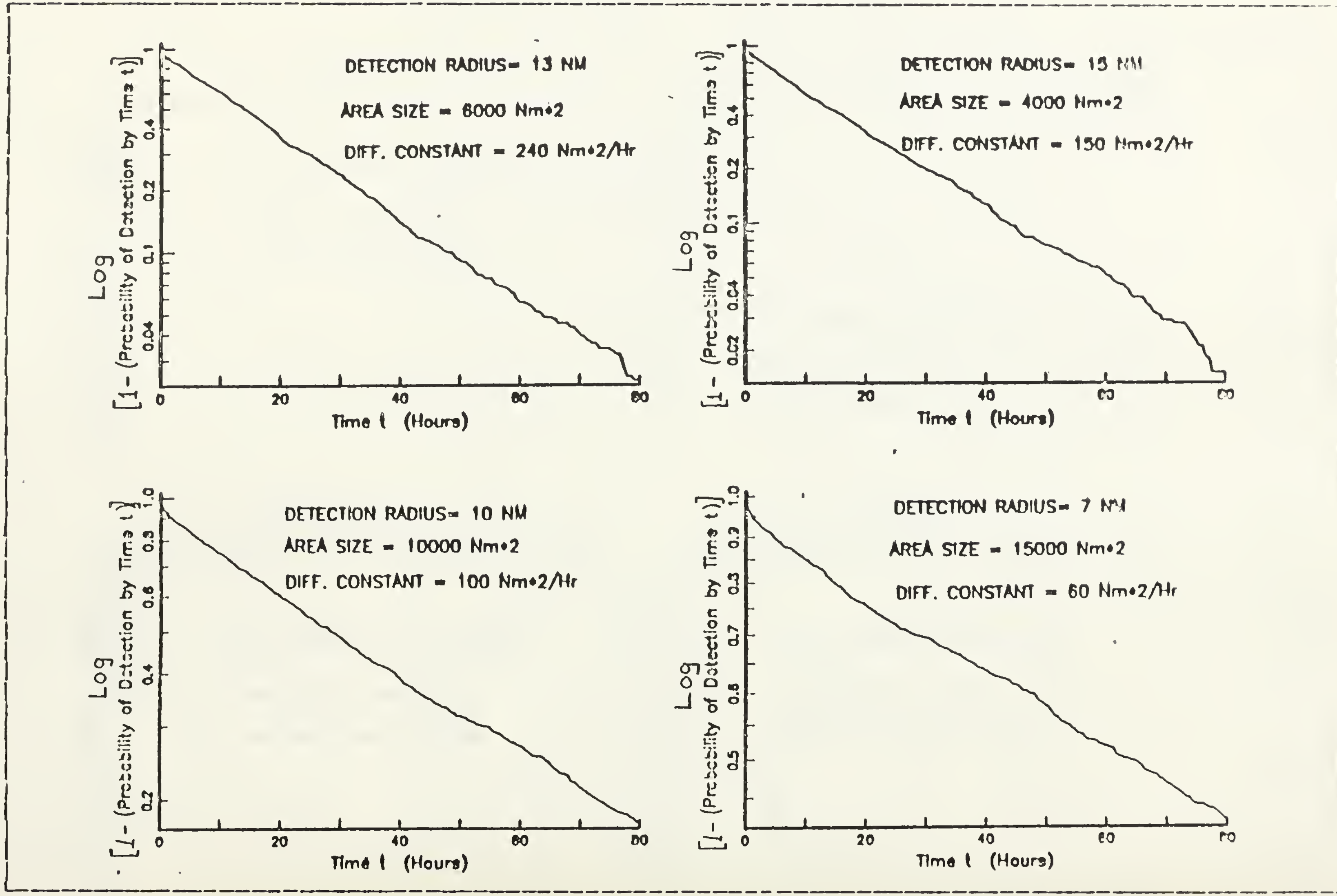


Figure 3.2 1-P(t) Plotted on Logarithmic Scale.

1. Submodel for α

In our model, we know that the target starting position is uniformly distributed over the search area A, and the searcher has a detection capability over a disk which covers an area πR^2 with probabilty 1. So, we may expect that at the beginning of the search, i.e., when t=0, detection probability will be equal to $\pi R^2/A$.

If we subsititute t=0 in equation (3.1), we get

$$P(0) = 1-\alpha$$

Then

$$P(t=0) = \frac{\pi R^2}{A} = 1-\alpha \qquad (3.2)$$

which implies that

$$\alpha = 1 - \frac{\pi R^2}{A} \qquad (3.3)$$

2. Submodel for β

This submodel will include all independent variables R, D, A and t, and will therefore be more complex. We will study each independent variable separately in order to simplify the problem (We will change one variable while holding the others fixed.) The relationship between β and these variables will be estimated. Eventually we will combine these submodels for a final submodel.

a. The Relationship Between Diffusing Constant D and β

For this case, area size A and radius R were held fixed at 10000 nm^2 10 nm. respectively. Diffusion constant D was varied between 20 nm^2/hr and 300 nm^2/hr The simulation results are displayed in Figures 3.3.

By using the least squares estimation method on ln[1-P(t)] data values, a best fit β was obtained for each diffusion constant D.

These β values are plotted against the corresponding D values in Figure 3.4. They fall approximately on a straight line. Again, by using the least square estimation method, the following linear equation was obtained

$$\beta = 0.00308 + 0.000205\ D \qquad (3.4)$$

$$\beta \approx 0.000205\ D$$

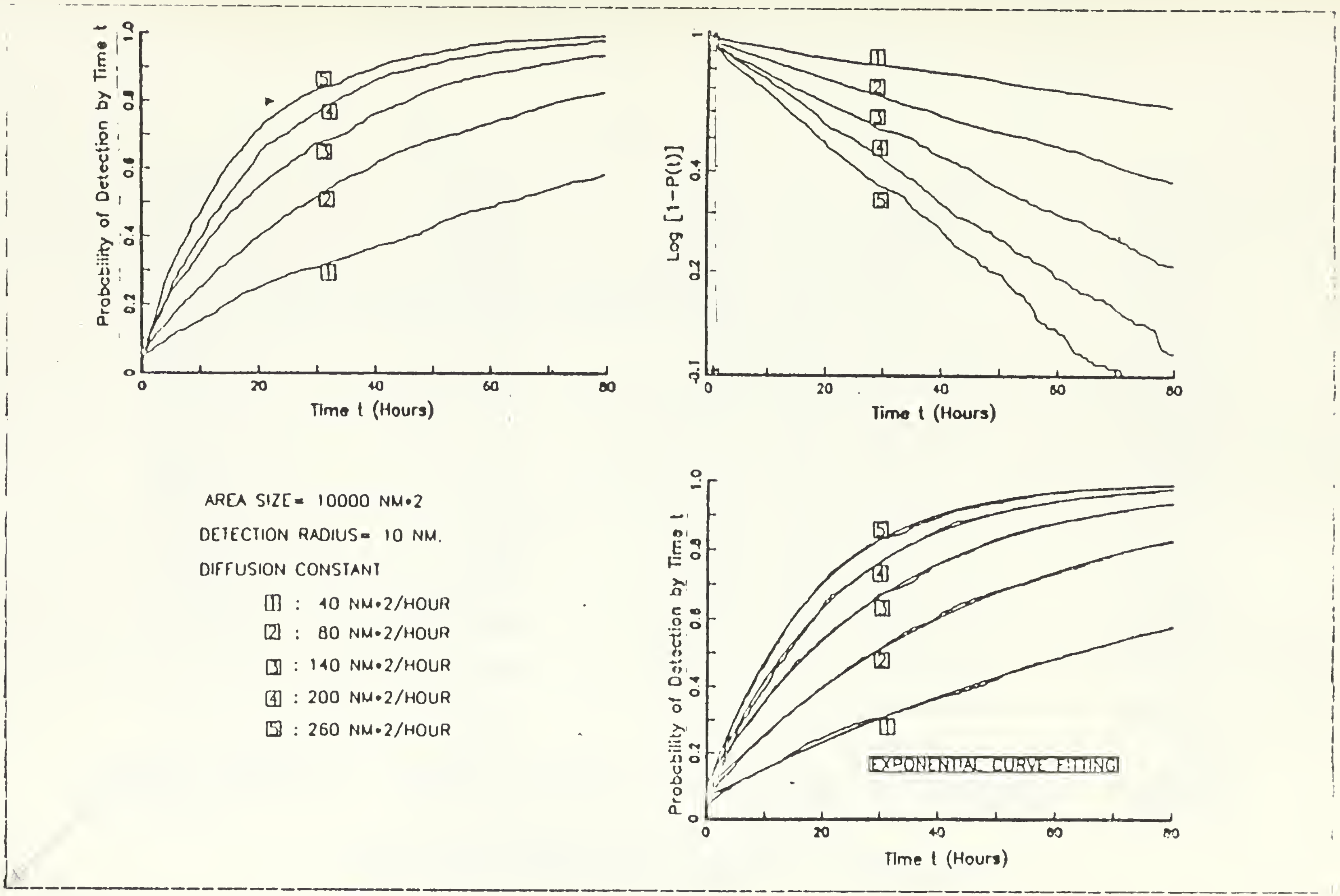


Figure 3.3 P(t) for 5 Diffusion Constants.

TABLE I

Exponential Curve Fitting for Different Diffusion Constants R=10 nm A=10000 nm^2

Diffusion Constant (nm^2/hr)	Theoretical α Value ($1-\pi R^2/A$)	Exponential Least Square Curve Fitting For $\alpha e^{-\beta}$	Estimated β Value
20	0.9686	0.940 EXP{-0.00503 T}	0.00503
40	0.9686	0.937 EXP{-0.00989 T}	0.00989
60	0.9686	0.912 EXP{-0.01523 T}	0.01523
80	0.9686	0.915 EXP{-0.02086 T}	0.02086
100	0.9686	0.981 EXP{-0.02615 T}	0.02615
120	0.9686	0.902 EXP{-0.02711 T}	0.02711
140	0.9686	0.897 EXP{-0.03341 T}	0.03341
160	0.9686	0.931 EXP{-0,03728 T}	0.03728
180	0.9636	0.917 EXP{-0.03953 T}	0.03953
200	0.9686	0.915 EXP{-0.04559 T}	0.04559
220	0.9686	0.946 EXP{-0.05026 T}	0.05026
240	0.9686	0.890 EXP{-0.04986 T}	0.04986
260	0.9686	0.909 EXP{-0.05583 T}	0.05583
280	0.9686	0.855 EXP{-0.06325 T}	0.06325
300	0.9686	0.850 EXP{-0.06239 T}	0.06239

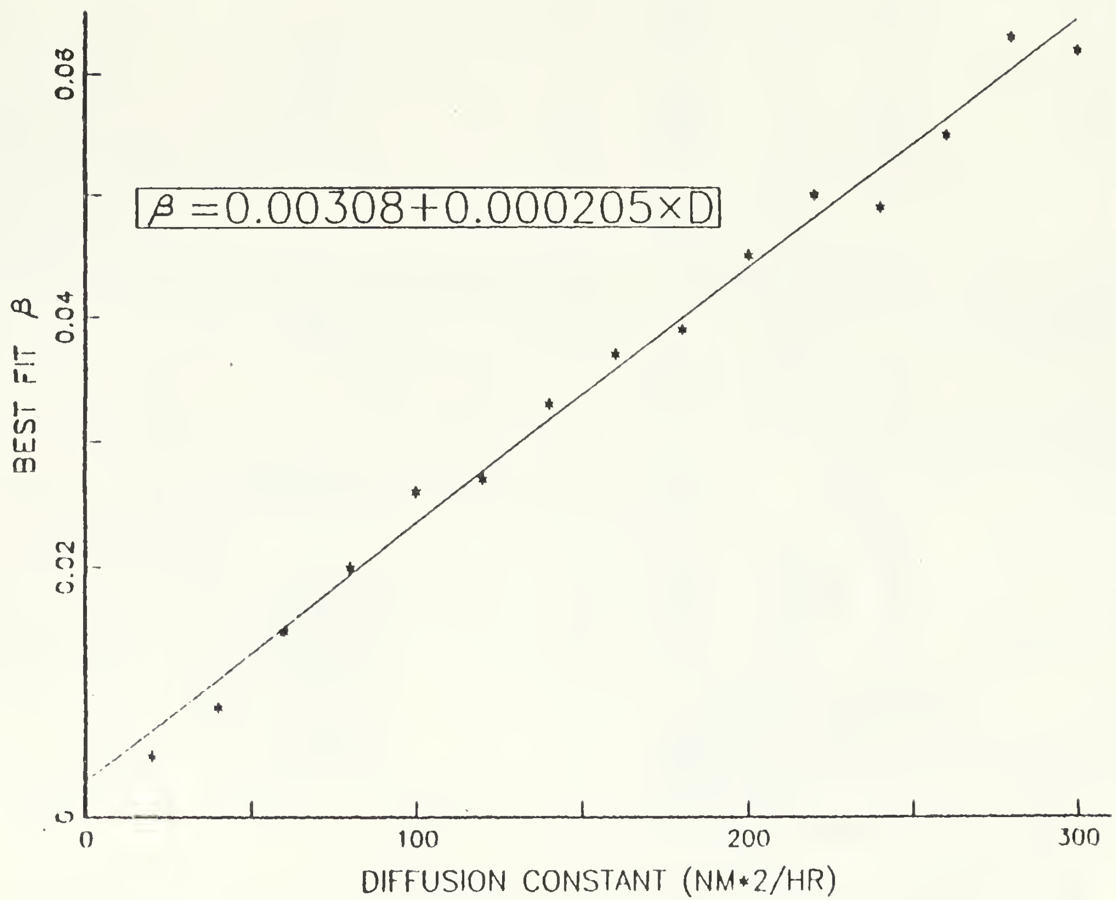


Figure 3.4 Best Fit β vs. Diffusion Constant.

b. The Relationship Between Area Size A and β

This time the diffusion constant D and detection radius R were held at 100 nm^2/hr and 10 nm respectively. The area size A was varied between 1000nm^2 and 20000 nm^2. The simulation results and the least square estimation results for the log transformed data values are displayed in Figure 3.5 and and Table II

Figure 3.6 shows a plot of the best fit β against area size A. These points fit very closely with the power function

$$\beta = 0.77A^{-1.49}$$

(A least squares procedure on the log transformed data was used to determine the values 0.77 and -1.49). To achieve more natural final units, the model is modified slightly as follows:

$$\beta = 0.77A^{-1.5} \qquad (3.5)$$

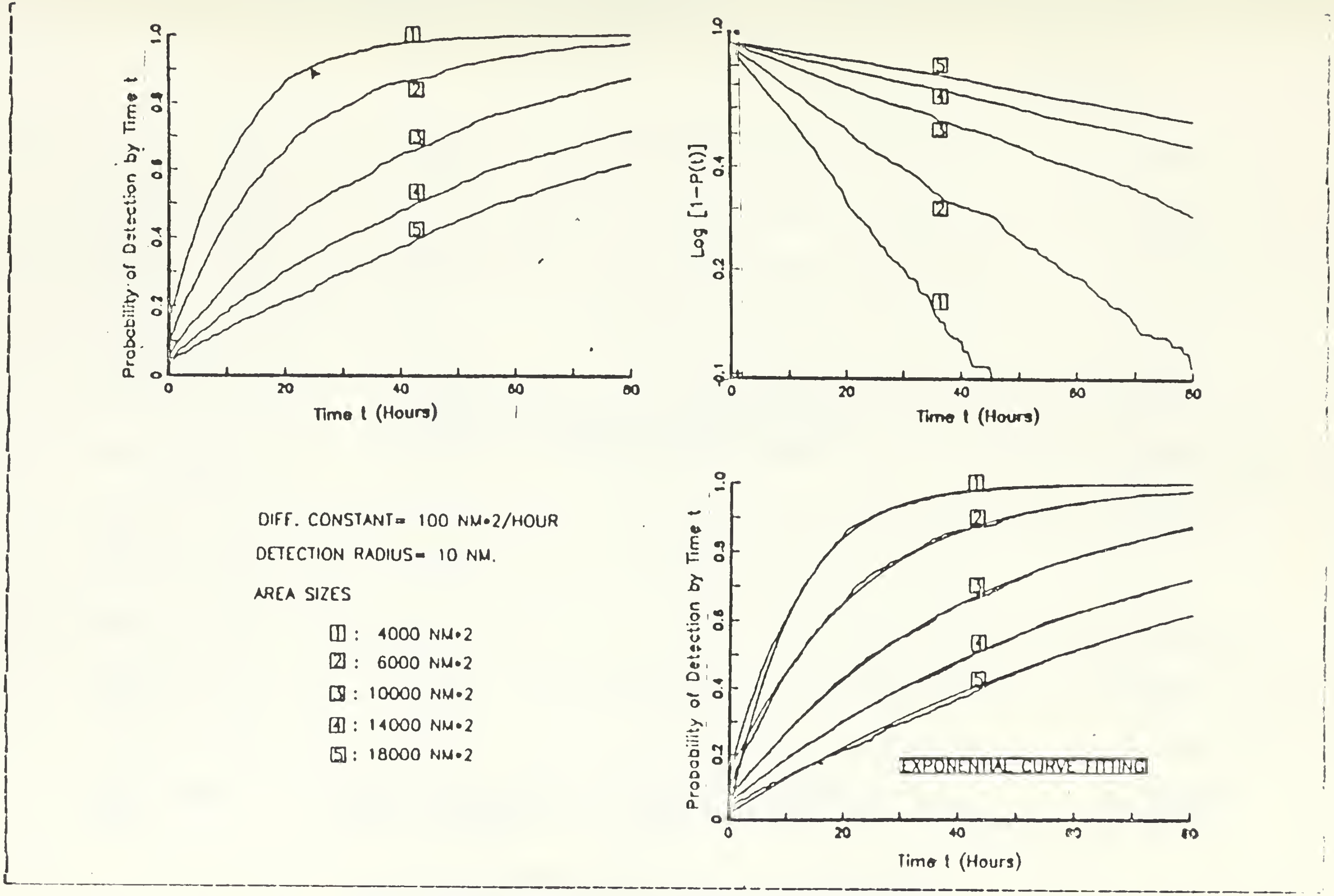


Figure 3.5 P(t) for 5 Different Area Value.

TABLE II

Exponential Curve Fitting For Different Area Sizes R=10 nm D=100 nm^2/hr

Area Size (nm^2)	Theoretical α Value $(1-\pi R^2/A)$	Exponential Least Square Curve Fitting for $\alpha e^{-\beta}$	Estimated β Value
1000	0.686	0.651 EXP{-0.91867 T}	0.91867
2000	0.843	0.831 EXP{-0.26581 T}	0.26581
4000	0.922	0.991 EXP{-0.09295 T}	0.09295
6000	0.948	0.858 EXP{-0.04521 T}	0.04521
8000	0.961	0.901 EXP{-0.03163 T}	0.03163
10000	0.968	0.939 EXP{-0.02469 T}	0.02469
12000	0.974	0.941 EXP{-0.01922 T}	0.01922
14000	0.978	0.951 EXP{-0.01537 T}	0.01537
16000	0.980	0.965 EXP{-0.01361 T}	0.01361
18000	0.983	0.984 EXP{-0.01183 T}	0.01183
20000	0.984	0.987 EXP{-0.01027 T}	0.01027

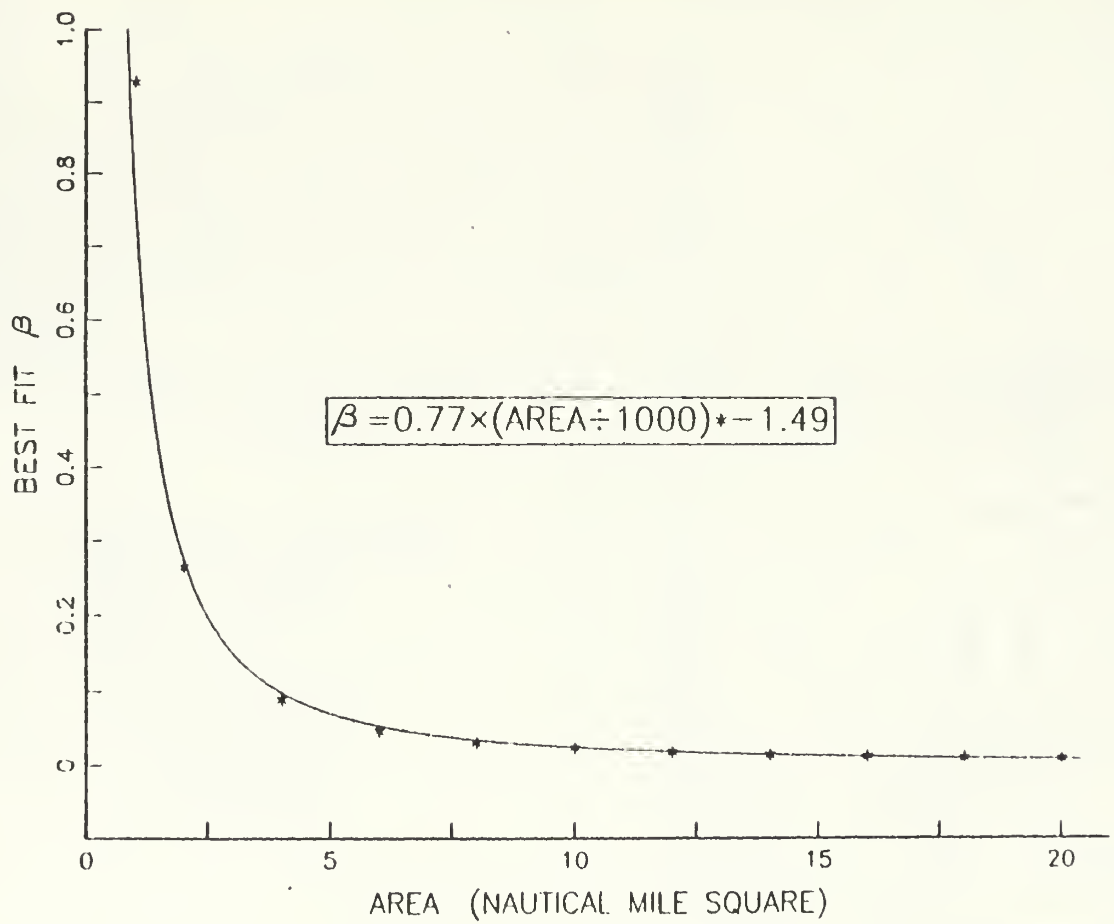


Figure 3.6 Best Fit β vs. to Area.

c. The Relationship Between Detection Radius R and β

Again, the same procedures were applied for this case. R was varied between 1 and 30 nm, while D and A were held fixed at 100 nm^2/hr and 10000 nm^2 respectively.

The simulation results and least square estimation results are displayed in Figure 3.7 and Table III. In Figure 3.8, the scatter plot of estimated β values shows a linear relationship between R and β.

Least square estimation for this line is

$$\beta = -0.00105 + 0.00278R \qquad (3.6)$$

$$\beta \approx 0.00278R$$

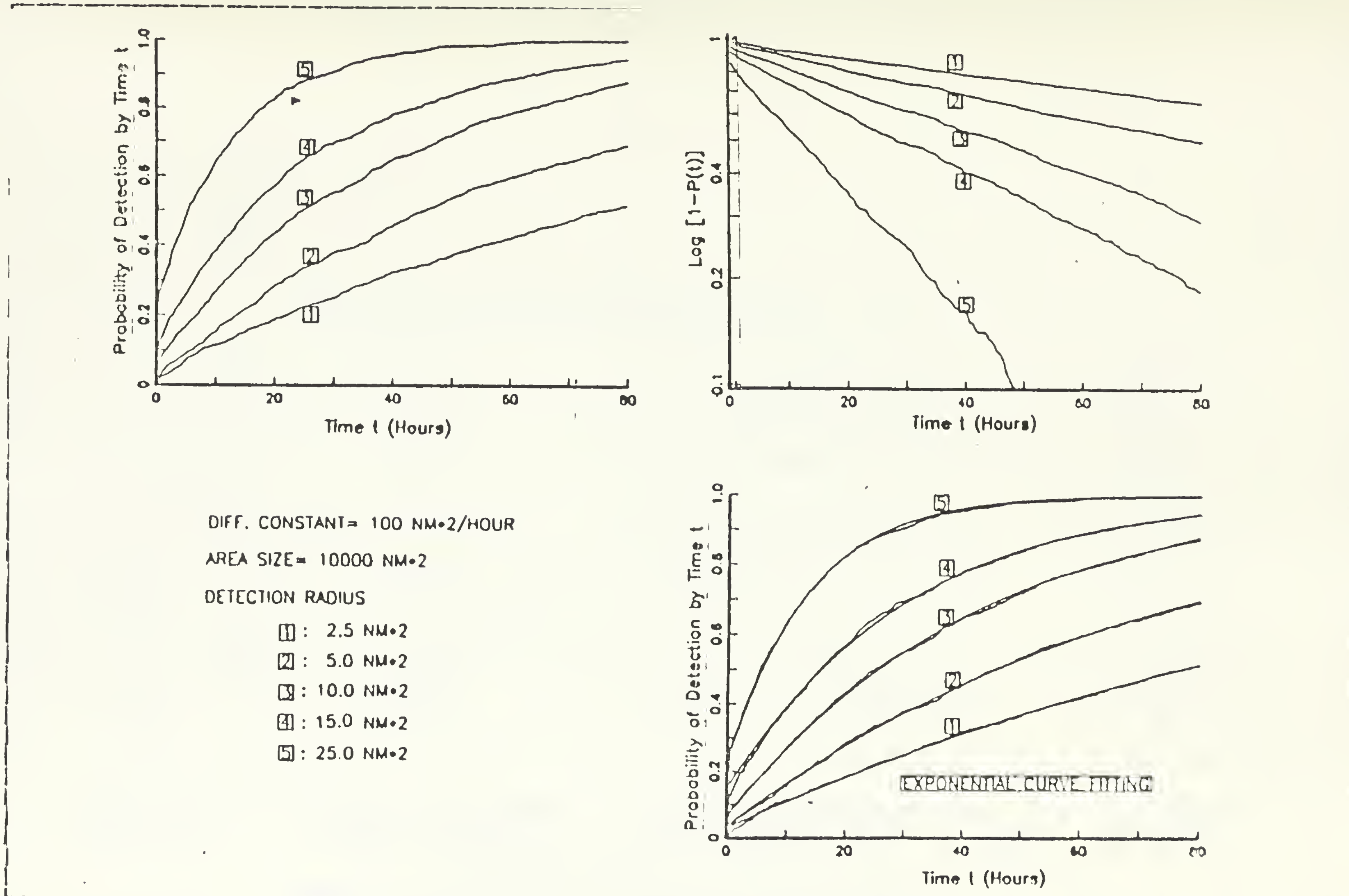


Figure 3.7 P(t) for 5 Radius Value.

TABLE III

Exponential Curve Fitting For Different Detection Radiuses $D=100\ nm^2/hr$ $A=10000\ nm^2$

Detection Radius (nm)	Theoretical α Value $(1-\pi R^2/A)$	Exponential Least Square Curve Fitting For $\alpha e^{-\beta}$	Estimated β Value
1.0	0.999	0.992 EXP{-0.00263 T}	0.00263
2.5	0.998	0.978 EXP{-0.00862 T}	0.00862
5.0	0.992	0.967 EXP{-0.01449 T}	0.01449
10.0	0.968	0.934 EXP{-0.02462 T}	0.02462
15.0	0.929	0.852 EXP{-0.03361 T}	0.03361
20.0	0.874	0.845 EXP{-0.05266 T}	0.05266
25.0	0.804	0.770 EXP{-0.07443 T}	0.07443
30.0	0.717	0.445 EXP{-0.08259 T}	0.08259

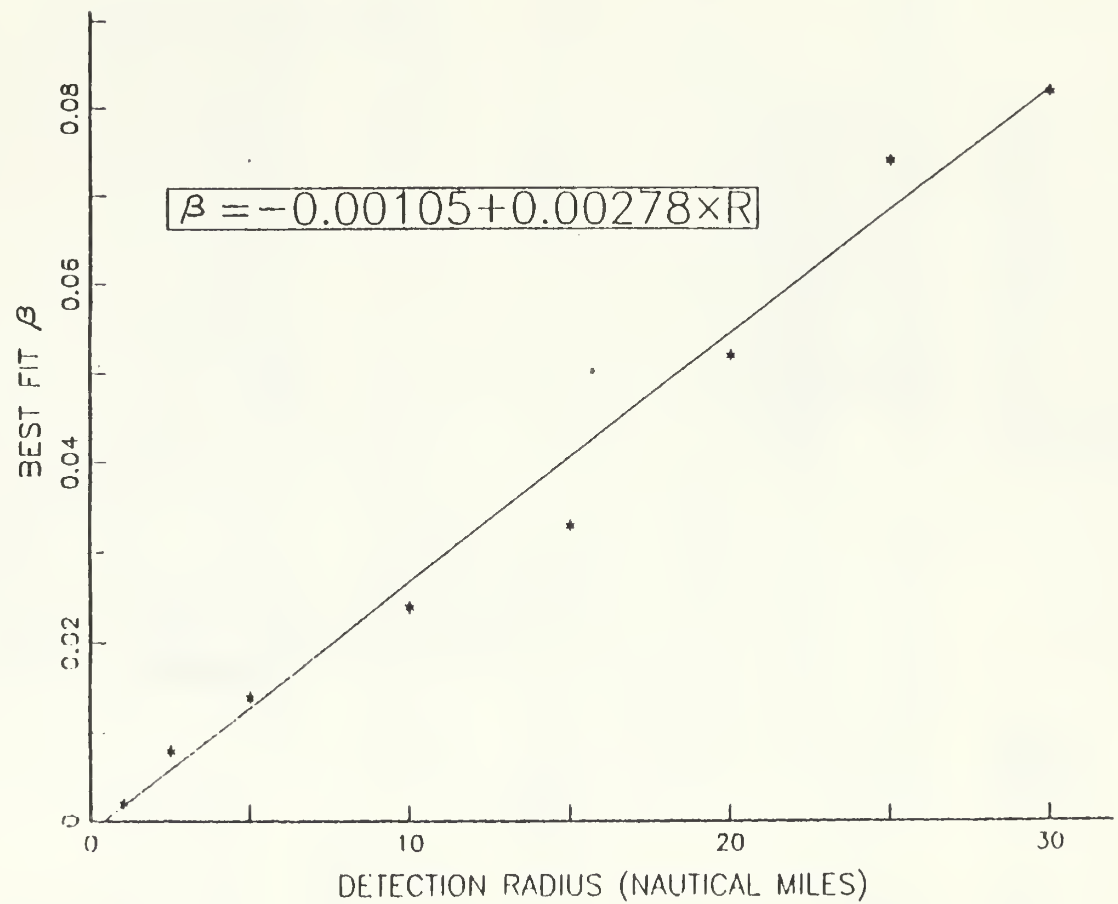


Figure 3.8 Best Fit β vs. Radius.

We can summarize these observations as follows:

$\beta \propto D$

$\beta \propto A^{-1.5}$

$\beta \propto R$

where "$\propto$" means "is proportional to" which suggests that

$$\beta = K \frac{RD}{A^{1.5}} \tag{3.7}$$

for the proper value of K. To estimate K is the final model building step.

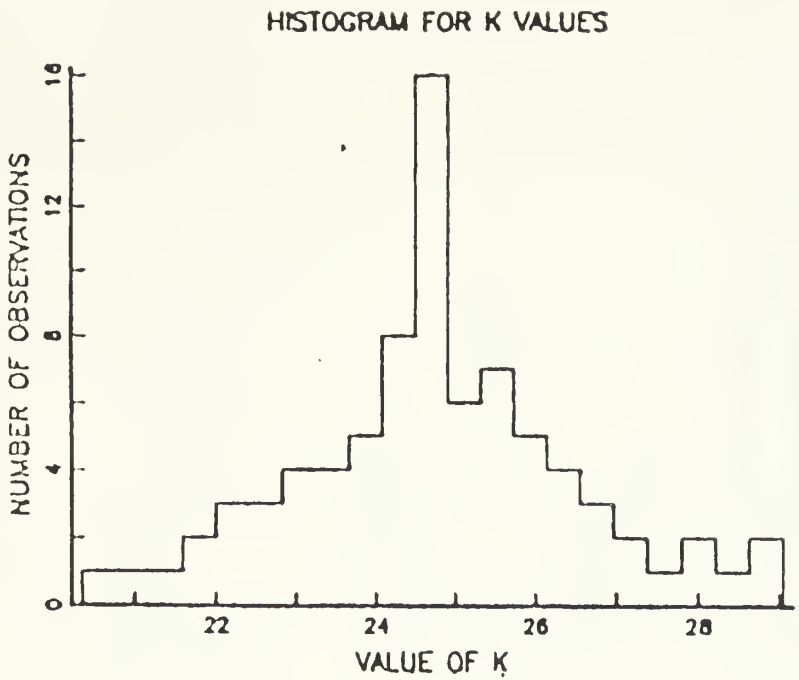
d. Estimation of the Coefficient K

We can calculate the K value for each simulation with the expression

$$K = \frac{\beta A^{1.5}}{RD} \tag{3.8}$$

where β is the "best fit" value for that simulation run. Then with these sample K values we may find best overall estimate.

In addition to 56 simulations already completed, 25 additional simulations were conducted to produce a total of 81 sample K values. The histogram and the statistical table values for this data are displayed in Figure 3.9.



X	: K
SELECTION	: ALL
X LABEL	: VALUE OF K
NO. OF ELEMENTS	: 81
X MEAN	: 24.7
STD. DEVIATION	: 1.73
SKEWNESS	: 0.104
KURTOSIS	: 0.136
5-PERCENTILE	: 21.8
25-PERCENTILE	: 23.9
MEDIAN	: 24.6
75-PERCENTILE	: 25.7
95-PERCENTILE	: 28
X MIN.	: 20.7 20.8 21.2
X MAX.	: 29 28.9 28.3

Figure 3.9 Observed Values of K.

If we recall the equation (3,1), (3.3) and (3.7)

$$P(t) = 1 - \alpha e^{-\beta t}$$

$$\alpha = 1 - \frac{\pi R^2}{A}$$

$$\beta = K \frac{RD}{A^{1.5}}$$

Substituting these α and β values in equation (3.1), we derived our final analytical model for first detection probabilities as follows

$$P(t) = 1 - \left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t} \qquad (3.9)$$

where P(t) refers to the probability of first detection occurs on or before time t.

D. VERIFICATION OF THE MODEL

1. Dimension Analysis

From equation (3.9) we see that

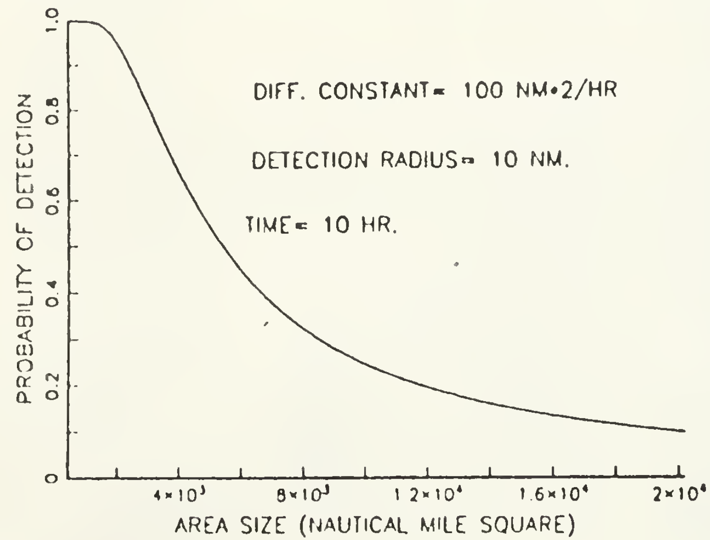
$$\frac{KRDt}{A^{1.5}}$$

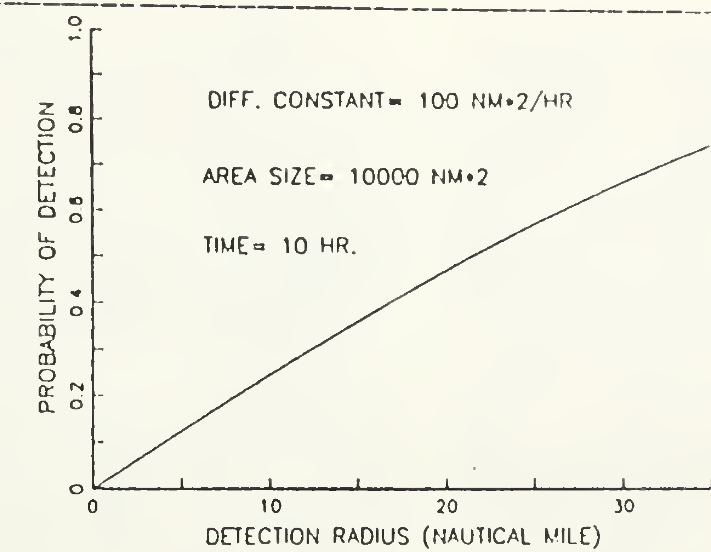
must be dimensionless. This implies that the coefficient K must be dimensionless. (If we had set the power of A to 1.49 versus 1.5, then the dimension of K would be $nm^{0.2}$, not a natural unit.)

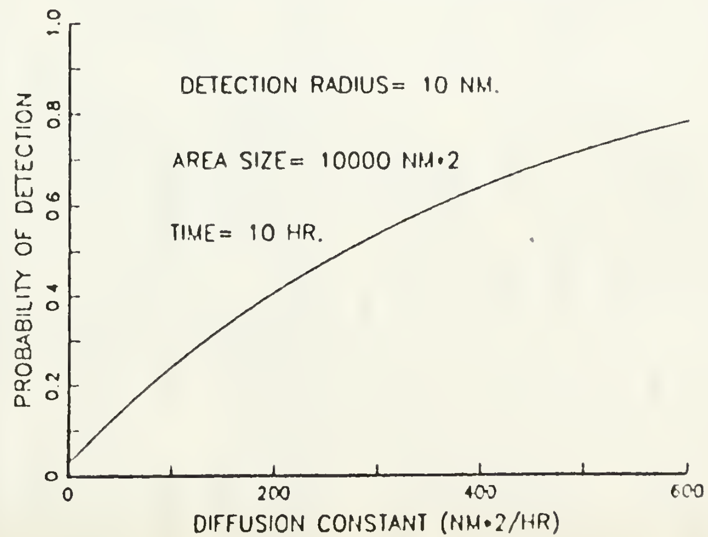
2. <u>Sensitivity Analysis for Independent Variables</u>

If we hold fixed all independent variables R,D,t and change the area size A, we will observe that as we increase the A, the probability of detection decreases and vice versa. This result is demonstrated in Figure 3.10. As we increase the size of the search area, more area will be available for the target to escape from the searcher. Therefore we may expect lower detection probabilities for larger search area sizes.

A similar sensitivity analysis is applied for the other independent variables R,D and t. If we look at the results displayed in Figure 3.10, we may observe that as we increase these variables, the detection probabilities increase simultaneously. These results seem reasonable, because as we increase the search time or detection radius, we may have more chanses to detect the target. Also an increase in the D value means that the target will travel more distance during any time interval and will thus be more likely to enter the detection disk.







P=PROBABILITY OF DETECTION

D=DIFFUSION CONSTANT (NM•2/HR)

R=DETECTION RADIUS (NM.)

A=AREA SIZE (NM•2)

T=TIME (HR.)

$$P = 1 - \left(1 - \frac{3.14 \times R}{A}\right) \times EXP^{-K \frac{R \times D}{A^{1.5}} \times T}$$

Figure 3.10 Sensitivity Analysis for A, D and R.

3. Final Verification

There was no actual data available from real life observations. Therefore, the output of DIFSIM is used for final verification of the model. For this purpose, combinations of the following independent variables were used both in our analytical model and as input to DIFSIM.

D= 40, 80, 140, 200 Nm*2/Hr

R= 2.5, 10, 20, 25 Nm

A= 4000, 8000, 12000, 16000 Nm*2

The outputs are displayed in Figure 3.,11, 3.12, 3.13. It is observed that the simulation and model probabilities are generally very close to each other. Only during early search hours do the simulation curves sometimes go above the model curve. This means that our model predicts fewer detection than the diffusion simulation model during the early search hours. Since this difference is at maximum .03 or .04, we conclude that the model provides a good fit, which gets better for larger time t. For small t,the model appears to underestimate the probability of detection.

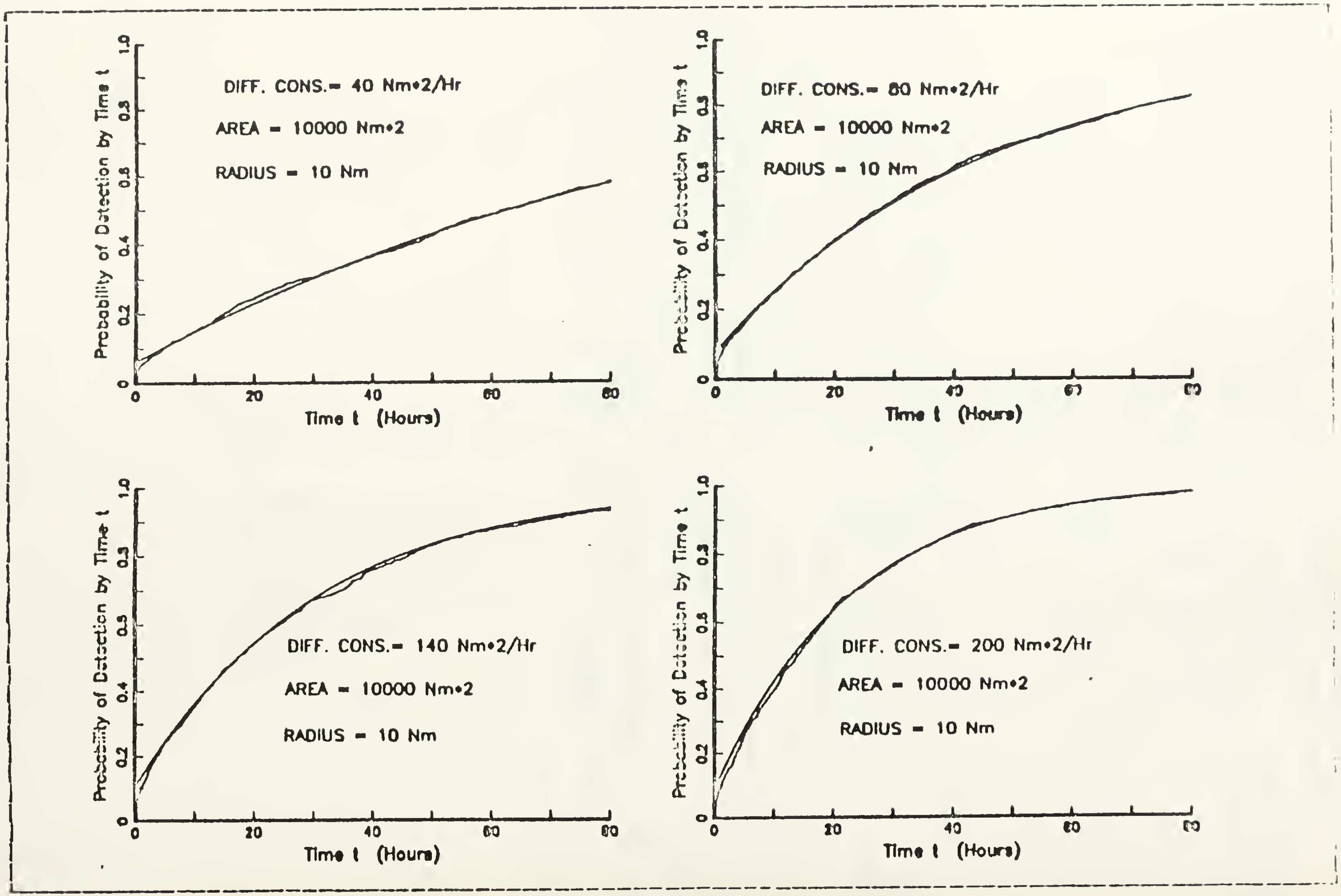


Figure 3.11 Model Verification Results (D Varies).

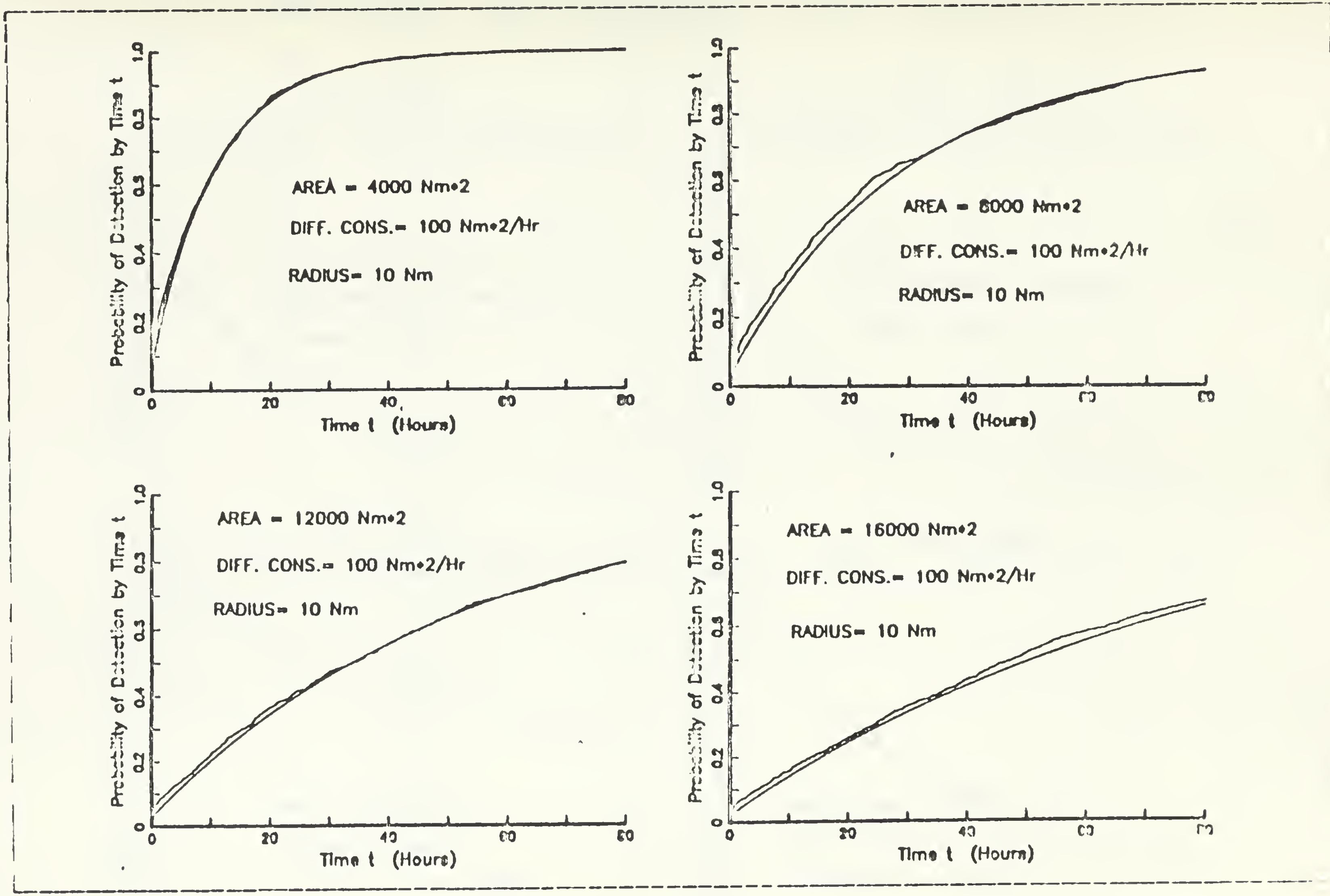


Figure 3.12 Model Verification Results (A Varies).

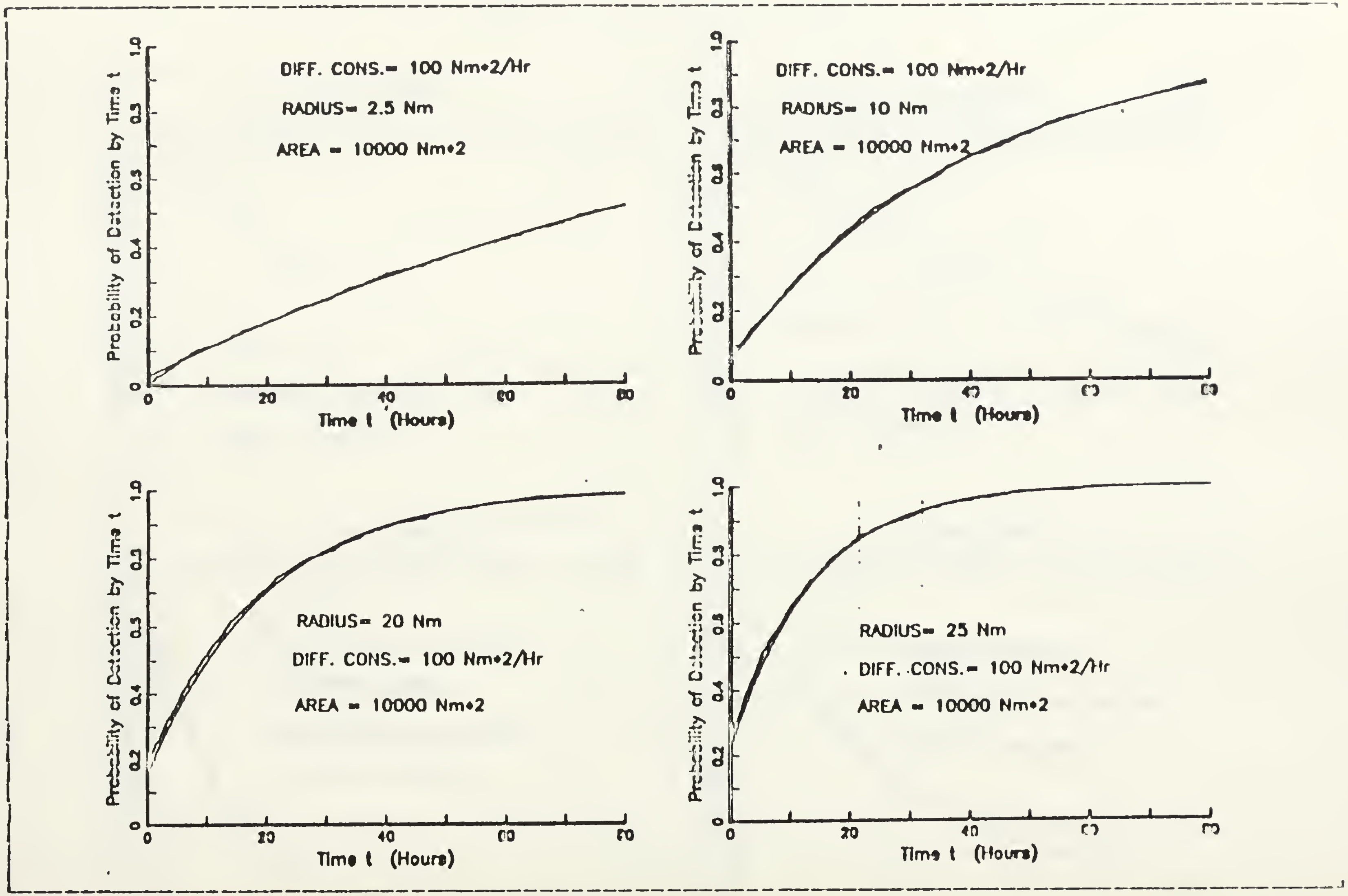


Figure 3.13 Model Verification Results (R Varies).

For the model presented here, the instantaneous rate of detection is the constant

$$\beta = K \frac{RD}{A^{1.5}}$$

The fact that the DIFSIM probability of detection by time t generally exceeds that predicted by the model suggests that the instantaneous probability of detection produced by DIFSIM exceeds (at least in early hours of the simulation). To test this hypothesis, DIFSIM was run with the parameters

D=100 nm^2/hr

R=10 nm

A=10000 nm^2

Then for each 5 hour period between 0 and 80 hours, the least squares best fit was obtained. (That is, the best fitting

$$1 - (1 - \frac{\pi R^2}{A}) e^{-\beta t}$$

was calculated). These values are plotted in Figure 3.14, and appear to approach from above the model value of

$$\frac{24.7 \times 100 \times 10}{10000} = .0247$$

Thus, in this case at least, it appears that the instantaneous probability of detection starts at some high value and decreases asymptotically to a steady state value given by

the model presented here. This observation remains to be proved in general.

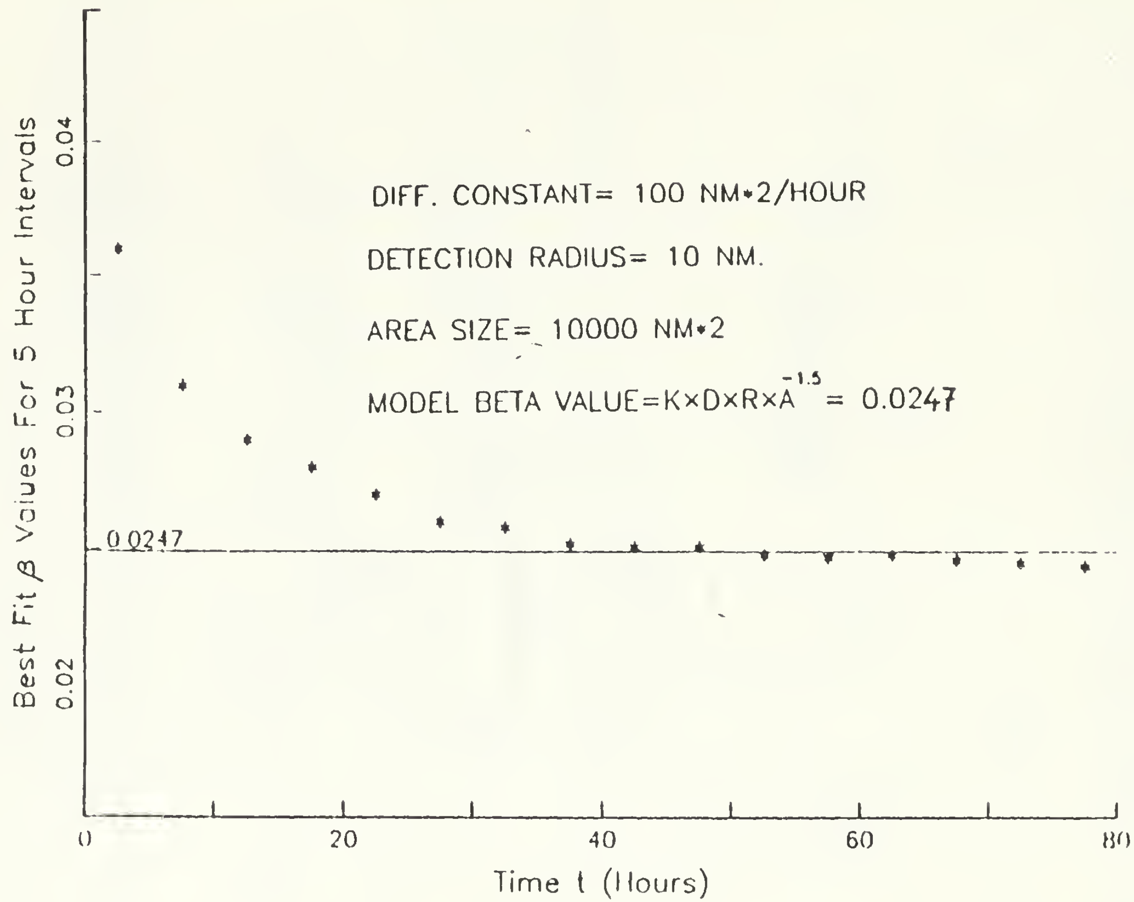


Figure 3.14 Instantaneous Detection Rate (β) vs. Time t.

IV. PROBABILISTIC ANALYSIS OF THE MODEL

A. CUMULATIVE DETECTION PROBABILITY FUNCTION

Since the following properties for general cumulative probability function hold for our mathematical model, equation (3.9), we may assume that this model also represents a cumulative density function (cdf) for first detection time t.

These properties are

1. $\lim_{t \to \infty} F(t) = 1$

$$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} \left[1 - \left(1 - \frac{\pi R^2}{A}\right) e^{-K\frac{RD}{A^{1.5}}t} \right]$$

$$= 1 - \left(1 - \frac{\pi R^2}{A}\right) . \, 0 = 1$$

2. F(t) is a nondecreasing function

If we take the the first derivate of F(t) with respect to t, we get

$$\frac{dF(t)}{dt} = \left(K \frac{RD}{A^{1.5}}\right)\left(1 - \frac{\pi R^2}{A}\right) e^{-K\frac{RD}{A^{1.5}}t} \geqslant 0$$

since this equation always has nonnegative values, we may assume that F(t) is a nondecreasing function.

3. F(t) is a continuous function.

Therefore we may define cdf as following.

$$F(t) = P(T \leq t) = P(\text{Detection up to time } t)$$

$$F(t) = 1 - \left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t} \qquad (4.1)$$

B. DETECTION PROBABILITY DENSITY FUNCTION

We can derive the detection probability density function (pdf), f(t), by taking the first derivative of cdf with respect to time t.

$$f(t) = \frac{dF(t)}{dt}$$

$$= \left(K \frac{RD}{A^{1.5}}\right)\left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t} \qquad (4.2)$$

If we integrate this function from 0 to ∞ we have

$$\int_0^\infty f(t)\,dt = \int_0^\infty \left(K \frac{RD}{A^{1.5}}\right)\left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t}\, dt$$

$$= \left(1 - \frac{\pi R^2}{A}\right) \left| \; e^{-K \frac{RD}{A^{1.5}} t} \; \right|_0^\infty$$

$$= \left(1 - \frac{\pi R^2}{A}\right) \neq 1$$

Since this integration is not equal to 1, the equation (4.2) doesn't represent a <u>proper</u> density function for t.

C. EXPECTED FIRST DETECTION TIME

Let $\bar{F}(t)$ denote the cumulative nondetection probability function (cndf).

$$\bar{F}(t) = P(\text{No detection up to time } t) = p(T>t)$$
$$= 1 - P(T \leqslant t) = 1 - F(t)$$

$$\bar{F}(t) = \left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t} \qquad (4.3)$$

The following formula can be used to find the expected detection time E[T].

$$E[T] = \int_0^\infty \bar{F}(t)\, dt \qquad (4.4)$$

If we substitute (4.3) in (4.4)

$$E[T] = \int_0^\infty \left[\left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t}\right] dt$$

$$= \frac{A^{1.5}}{KRD}\left(1 - \frac{\pi R^2}{A}\right) \int_0^\infty \frac{KRD}{A^{1.5}} e^{-K \frac{RD}{A^{1.5}} t}\, dt$$

$$= \frac{A^{1.5}}{KRD}\left(1 - \frac{\pi R^2}{A}\right) \left| -e^{-K \frac{RD}{A^{1.5}} t} \right|_0^\infty$$

$$= \frac{A^{1.5}}{KRD}\left(1 - \frac{\pi R^2}{A}\right)$$

$$E[T] = \frac{A^{1.5}}{KRD} - \frac{A^{.5} \pi R}{KD} \qquad (4.5)$$

Expected first detection times are displayed for different diffusion constants (D), area sizes (A) and detection radiuses (R) in Figure 4.1.

D. CONDITIONAL DETECTION PROBABILITY FUNCTIONS

If we assume that there will be no detection at the beginning of the search period, we may derive the following conditional cdf.

$F_0(t) = P(\text{Detection up to time } t/\text{no det. at time } 0)$

$$= P[T \leqslant t / T > 0] = \frac{P[T>0, T \leqslant t]}{P[T>0]}$$

$$= \frac{P[0 < T \leqslant t]}{P[T>0]} \qquad (4.6)$$

If we substitute t=0 in equation (4.3), we get

$$P(T>0) = \bar{F}(0) = \left(1 - \frac{\pi R^2}{A}\right) \qquad (4.7)$$

and

$$P(0<T\leqslant t) = F(t) - P(t=0)$$

$$= \left[1 - \left(1 - \frac{\pi R^2}{A}\right) e^{-K \frac{RD}{A^{1.5}} t}\right] - \frac{\pi R^2}{A} \qquad (4.8)$$

By using (4.7) and (4.8) as a dominator and numerator in the equation (4.6), we have

$$F_0(t) = 1 - e^{-K \frac{RD}{A^{1.5}} t} \qquad (4.9)$$

This function (4.9) represents a cumulative probability function of an exponential distribution with parameter $K\frac{RD}{A^{1.5}}$. By using this fact, conditional expected first detection time $E[T_0]$ can be defined as follows:

$$E[T_0]=E[T/\text{No detection at time } 0]=\frac{A^{1.5}}{KRD} \qquad (4.10)$$

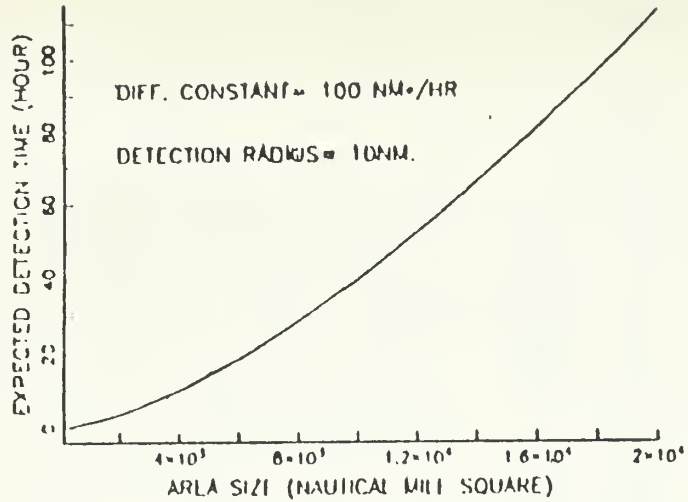
Also, we can write conditional detection probability density function $f_0(t)$ in the following form,

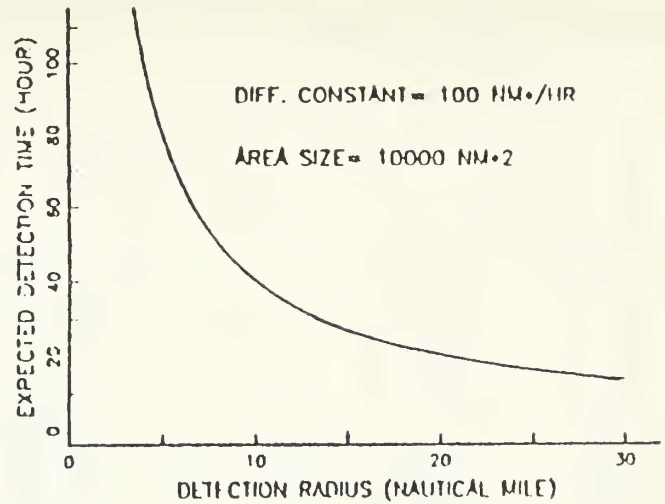
$$f_0(t)=K\frac{RD}{A^{1.5}}\,e^{-K\frac{RD}{A^{1.5}}t} \qquad (4.11)$$

If we compare equations (4.5) and (4.10), we will observe that,

$$\frac{A^{1.5}}{KRD}-\frac{A^{.5}\pi R}{KD} \leq \frac{A^{1.5}}{KRD}$$

so $E[T]\leq E[T_0]$. This inequality means that the conditional first detection time is greater than the unconditional first detection time. We can get this conclusion intuitively by thinking that we have an opportunity to detect the target at the beginning of the search period in the unconditional case. Therefore, for the unconditional case, we may expect an earlier first detection time than would be possible for the conditional case.





E[T]=EXPECTED DETECTION TIME (HR.)

D=DIFFUSION CONSTANT(NM•2/HR)

R=DETECTION RADIUS(NM)

A=AREA SIZE(NM•2)

$$E[T]=\frac{A^{1.5}}{25\times R\times D}-\frac{A^{.5}\times R\times 3.14}{25\times D}$$

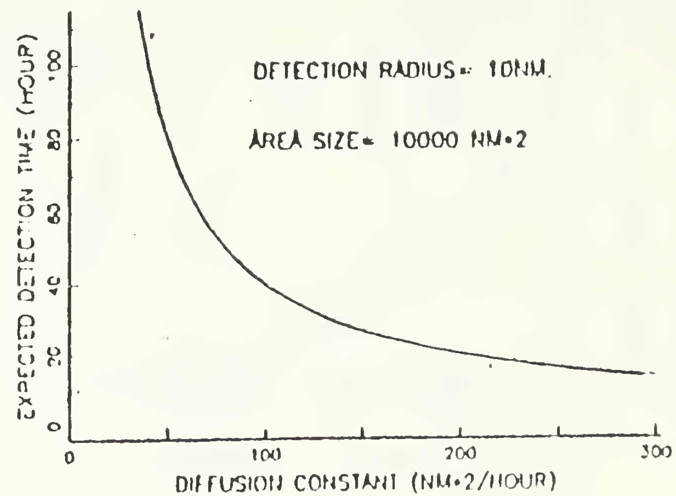


Figure 4.1 Expected First Detection Time.

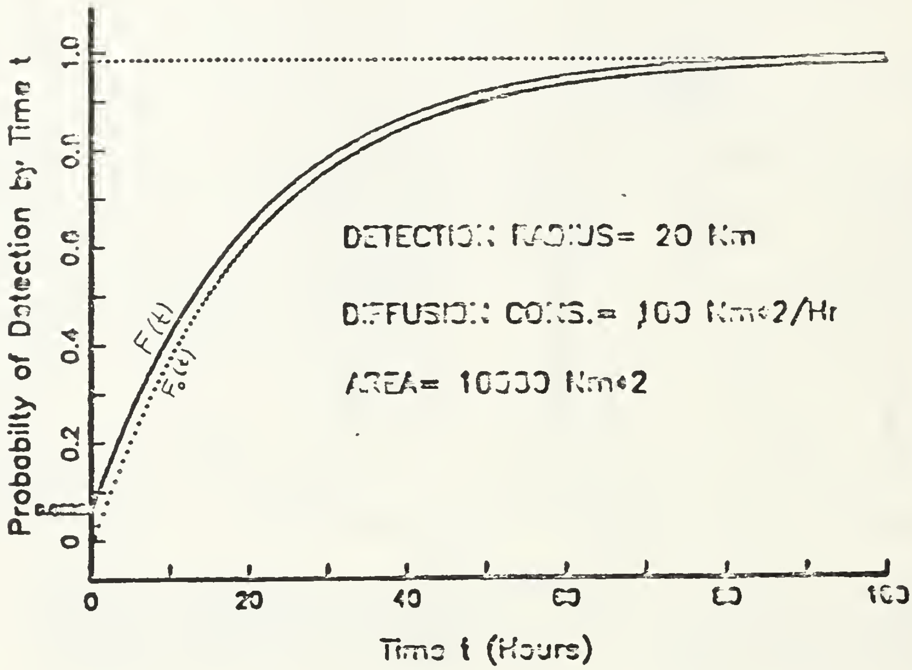


Figure 4.2 Comparison of $F(t)$ and $F_o(t)$.

V. THE RESULTS AND APPLICATIONS OF THE MODEL

A. EXPECTED DETECTION TIME FOR RANDOM TOUR MODEL

As we showed in Chapter II, a diffusion target gives an upper bound detection probability for the "equivalent" random tour model. Therefore, the diffusion model expected detection time, which is estimated by equation (4.5) should be a lower bound for random tour expected detection time. That is, for

$E[T_D]$=Diffusion Model Expected Detection Time, and

$E[T_R]$=Random Tour Model Expected Detection Time, we have

$$E[T_D]=\frac{A^{1.5}}{KRD}-\frac{A^{.5}\pi R}{KD} \le E[T_R]$$

B. ONE-DIMENSIONAL DIFFUSION MODEL

In this thesis, two-dimensional diffusion motion was the basis for our model. The exponential type curves were used to estimate this model's outputs. In addition to this model, the one-dimensional diffusion model is simulated by computer program DIFSIM1. In this model, the target moves on a line segment L, according to diffusion constant D. The target's starting position is selected uniformly over this line segment L. Detection occurs whenever the target hits the designated end point. The target reflects off the other end point of line segment.

The results of different simulation results are displayed in Figure 5.1. Exponential curves, which were obtained by using the least squares estimation method, were

used to estimate the outputs of one-dimensional diffusion model as we did for two-dimensional case.

So we may expect that, for three or more dimensional diffusion models, we may use exponential type curves which are generated by a different set of parameters. For three-dimensional case, these parameters can be defined as follows:

V=Volume of the cubical search space.

R=Radius of the cookie-cutter detection sphere.

D=Diffusion constant.

T=Detection time.

C. APPLICATIONS OF THE MODEL

Our model can also be used to estimate the final detection probability of a system which includes more than one independent sensor. As an example, we may use the following scenario:

We want to use n sonobouys in order to detect a diffusing target in an area A. Each sonobouy has a cookie_cutter detection capability over a disk with radius R_i. Each sonobouy will be independetly located on the center of a square subsearch area A_i and operated for a time period t. If we make the following assumptions, we may use equation (3.9) to estimate the overall detection probability of this sonobouy pattern at the end of the search period t.

$$R = \left(\sum_{i=1}^{n} R_i^{\,2}\right)^{1/2}$$

$$A = \sum_{i=1}^{n} A_i$$

where R is the effective detection radius.

This formula also gives us an upper bound detection probability for a random tour model which moves in the same system.

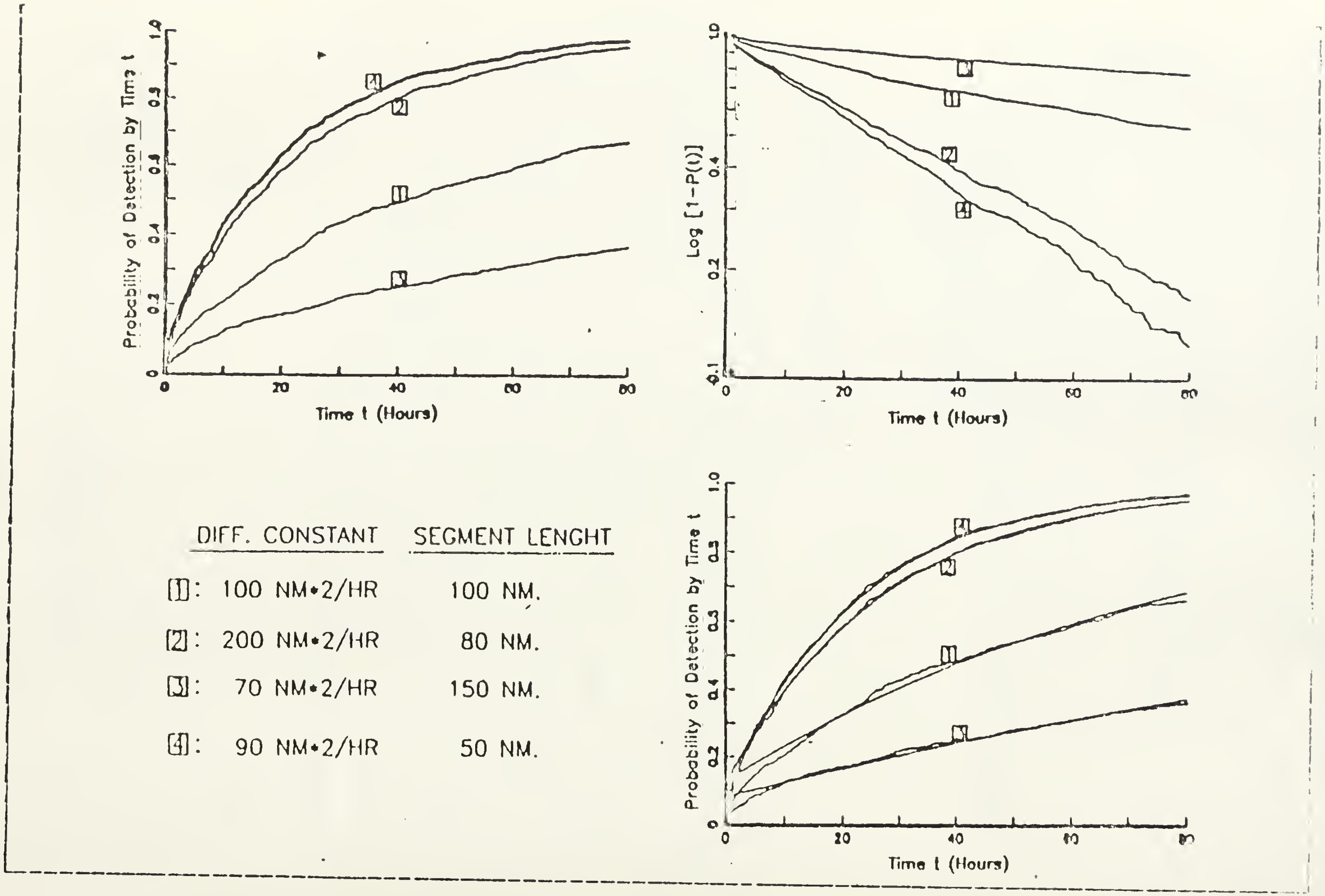


Figure 5.1 One-Dimensional Diffusion Model.

APPENDIX A
DIFSIM COMPUTER PROGRAM

In order to give access to the logic used in building the simulation models DIFSIM and DIFSIM1, a complete program listing is included in this Appendix following the list of variables used in the simulation models.

LIST OF VARIABLES

REP =Number of replications.

MAX =Detection period as a minute.

R =Radius of detection disk in nautical miles.

DIF =Diffusion constant D in square nautical miles per hour.

SIDE=The length of the square search area side in nautical miles.

AREA=Area size A in square nautical miles.

INC =Time increment t for each discrete step in minutes.

PROB=Probability of detection

POSX=X component of target position.

POSY=Y component of target position.

DIST=The distance between the target location and the center of the detection disk.

IX1 =Seed number for uniform random number.

Ix2 =Seed number for standard normal random variable.

```
C*******************************************************************
C*  THIS PROGRAM SIMULATES 2 DIMENSIONAL DIFFUSION
C*                  TARGET DETECTION MODEL.
C*******************************************************************
      REAL XS(2000),YS(2000),XT(8000),YT(8000),TI...(2     ),
     *PROB(2000),MAX,INC,NUM(8000),TI(2000)
      INTEGER REP,CONTR,MAXCTR ,CONTI
C  DEFINE THE INPUT PARAMETERS R,D A,INC,REP AND MAX.
C  ******************* INPUT SECTION ***************
      REP=1500
      MAX=150.*60.
      R=15.**2
      DIF=100.
      AREA=10000
      SIDE=AREA**0.5
      SID2=SIDE*2.
      INC=5.
      STRDV=SQRT(INC*DIF/60.)
      MAXCTR=INT(MAX/INC)+1
      DO 10 I=1,MAXCTR
        NUM(I)=0.
10    CONTINUE
      SER=SIDE/2
C  UNIFORM RANDOM VARIABLES ARE GENERATED FOR
C  STARTING POSITION OF THE TARGET.
      IX1=61899
      IY1=112233
      CALL LRND (IX1,XS,REP,1,0)
      CALL LRND (IY1,YS,REP,1,0)
      DO 500 I=1,REP
C  STARTING POSITION OF THE TARGET
             POSX=XS(I)*SIDE
             POSY=YS(I)*SIDE
             TIME=0.
             CONTR=1
C  STANDART NORMAL RANDOM VARIABLES ARE GENIRATED
C  TO DETERMINE THE NEXT LOCATION OF THE TARGET.
             IX2=IX1*I
             IY2=IY1*I
             CALL LNORM (IX2,XT,8000,1,0)
             CALL LNORM (IY2,YT,8000,1,0)
 200         DIST=((POSX-SER)**2)+((POSY-SER)**2)
C  CHECK FOR DETECTION.
             IF (DIST.LT.R) GO TO 300
             IF (CONTR.GT.MAXCTR) GO TO 350
C  TARGETS NEW LOCATION.
                TEMX=STRDV*XT(CONTR)+POSX
                TEMY=STRDV*YT(CONTR)+POSY
C  CHECK FOR REFLECTION.
                IF (TEMX.GT.SIDE)  TEMX= SID2-TEMX
                IF (TEMY.GT.SIDE)TEMY=SID2-TEMY
                IF (TEMX.LE.0.)  TEMX=-TEMX
                IF (TEMY.LE.0.)  TEMY=-TEMY
                  POSX=TEMX
                  POSY=TEMY
                TIME=TIME+INC
                CONTR=CONTR+1
                GO TO 200
300          CONTI=INT(TIME/INC)+1
             NUM(CONTI)=NUM(CONTI)+1.
350          CONTINUE
500   CONTINUE
```

```
      PROB(1)=NUM(1)/REP
      TI(1)=0.
      DO 700 I=2,MAXCTR
         TI(I)=FLOAT(I-1)*(INC/60.)
         II=I-1
C  COMPUTE DETECTION PROBABILITIES FOR EACH TIME INCREMENT.
         PROB(I)=PROB(II)+NUM(I)/REP
700   CONTINUE
      DO 750 I=1,MAXCTR,4
         I2=I+1
         I3=I2+1
         I4=I3+1
         WRITE (6,800) TI(I),PROB(I),TI(I2),PROB(I2),TI(I3)
     *,PROB(I3),TI(I4),PROB(I4)
800      FORMAT(2X,F6.2,2X,F7.5,3X,F6.2,2X,F7.5,3X,F6.2,2X
     *,F7.5,3X,F6.2,2X,F7.5)
750   CONTINUE
      STOP
      END
```

```
C****************************************************************
C* THIS PROGRAM SIMULATES 1 DIMENSIONAL DIFFUSION
C*                 TARGET DETECTION MODEL.
C****************************************************************
      REAL XS(2000),XT(8000),TIMDET(2000),TI(2000)
     *MAX,INC,PROB(2000),NUM(8000)
      INTEGER REP,CONTR,MAXCTR ,CONT1
C  DEFINE INPUT PARAMETERS D,D,SIDE,INC,REP AND MAX
C************* INPUT SECTION ****************
      REP=1000
      MAX=30.*60.
      DIF= 90.
      SIDE= 50.
      INC=5.
      STRDV=SQRT(INC*DIF/60.)
      MAXCTR=INT(MAX/INC)+1
      DO 10 I=1,MAXCTR
         NUM(I)=0.
10    CONTINUE
C UNIFORM RANDOM VARIABLES ARE GENERATED FOR
C STARTING POSITION OF THE TARGET.
      IX1=61899
      CALL LRND (IX1,XS,REP,1,0)
      DO 500 I=1,REP
C STARTING POSITION OF THE TARGET.
            POSX=XS(I)*SIDE
            TIME=0.
            CONTR=1
C STANDART NORMAL RANDOM VARIABLES ARE GENERATED
C TO DETERMINE THE NEXT LOCATION OF THE TARGET.
            IX2=IX1+I
            CALL LNORM (IX2,XT,8000,1,0)
200         IF (POSX.GE.SIDE) GO TO 300
C CHECK FOR DETECTION.
            IF (CONTR.GT.MAXCTR) GO TO 350
C TARGETS NEW LOCATION
                TEMX=STRDV*XT(CONTR)+POSX
C CHECK FOR REFLECTION.
                IF (TEMX.LE.0.) TEMX=-TEMX
                  POSX=TEMX
                TIME=TIME+INC
                CONTR=CONTR+1
                GO TO 200
300         CONT1=INT(TIME/INC)+1
            NUM(CONT1)=NUM(CONT1)+1.
350         CONTINUE
500   CONTINUE
      PROB(1)=NUM(1)/REP
      TI(1)=0.
      DO 700 I=2,MAXCTR
         TI(I)=FLOAT(I-1)*(INC/60.)
         II=I-1
C COMPUTE DETECTION PROBABILITIES FOR EACH TIME INCREMENT.
         PROB(I)=PROB(II)+NUM(I)/REP
700   CONTINUE
      DO 750 I=1,MAXCTR,4
         I2=I+1
         I3=I2+1
         I4=I3+1
         WRITE (6,900) TI(I),PROB(I),TI(I2),PROB(I2)
     *,TI(I3),PROB(I3),TI(I4),PROB(I4)
```

```
800       FORMAT(2X,F6.2,2X,F7.5,3X,F6.2,2X,F7.5,3X,F6.2,2X
     *,F7.5,3X,F6.2,2X,F7.5)
750   CONTINUE
      STOP
      END
```

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Thesis
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c.1 A mathematical model
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